Phase-Field Modeling of Crack Propagation in Anisotropic Media

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We study crack propagation in anisotropic media with a phase-field model where generalizations of the Principle of Local Symmetry (PLS) are not obvious. Our starting point is an extension of the PLS based on the analysis of forces acting on the crack tip [1,2]. When the crack kinks of a small angle $\delta\theta$ from its local axis, a torque

$G_\theta = \lim_{\delta\theta\to 0}[G(\theta+\delta\theta)-G(\theta)]/\delta\theta$

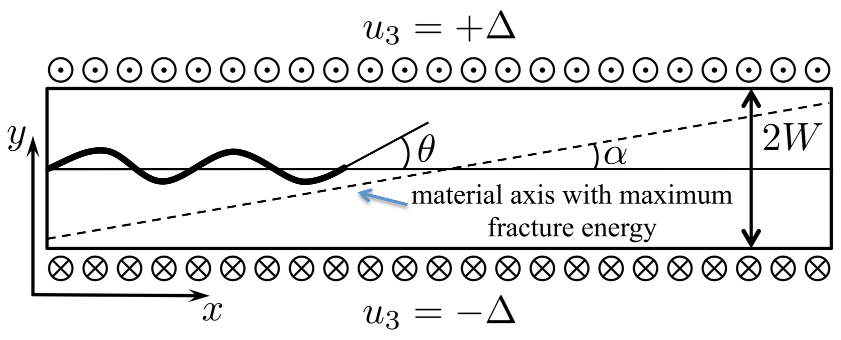
turns the crack tip in a direction that maximizes the energy release rate $G(\theta)$. For a fracture energy that depends on the propagation direction

$$\frac{d\Gamma_c(\theta)}{d\theta} = \Gamma_{c\theta} = 2\gamma_\theta \quad \xrightarrow{\text{torque balance condition}} \quad G_\theta = \Gamma_{c\theta}$$

Experimental motivation:

Recent tearing experiments of bi-oriented brittle thin sheets revealed that strong anisotropy suppresses straight crack propagation along so-called forbidden propagation directions, leading to crack paths with a characteristic sawtooth pattern [3]. Takei *et al.* proposed a criterion for the identification of forbidden propagation directions, that is $\Gamma_c+\Gamma_{c\theta\theta} < 0$. Although we study crack propagation for pure antiplane shear (thick samples), our analysis provides insights on the origin of oscillatory cracks in anisotropic media.

Phase-field model of crack propagation in anisotropic materials under mode III load



The phase field variable is a measure of the local damage of the material [4-6]: $\phi = 1$ intact, $\phi = 0$ broken

$$E = \iint d\vec{x}\left\{\mathcal{E}_{pf} + g(\phi)\left[\frac{\mu}{2}|\nabla u_3|^2 - \mathcal{E}_c\right] + \mathcal{E}_c\right\}$$

softening of the material $g(\phi) = 4\phi^3 - 3\phi^4$; energy threshold to start breaking the material

process zone size $\xi = \sqrt{\kappa/2\mathcal{E}_c}$

anisotropic surface energy:

$\mathcal{E}_{pf} = \frac{\kappa}{2}[(1+\epsilon\sin^2\alpha)(\partial_x\phi)^2 + (1+\epsilon\cos^2\alpha)(\partial_y\phi)^2 - \epsilon\sin(2\alpha)\partial_x\phi\partial_y\phi]$

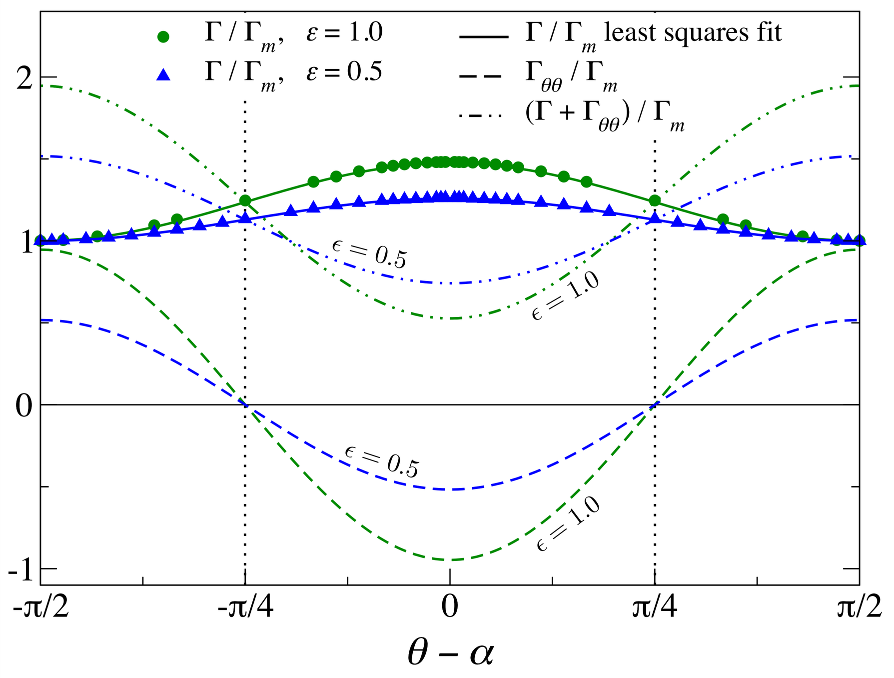
fracture energy along the direction of minimal surface energy

$\Gamma_m = 2\sqrt{2\kappa\mathcal{E}_c}\int_0^1 d\phi\sqrt{1-g(\phi)} = 2.027\sqrt{\kappa\mathcal{E}_c}$

quasistatic equilibrium condition $0 = \frac{\delta E}{\delta u_3}$

equations of motion of the phase field $\chi^{-1}\partial_t\phi = -\frac{\delta E}{\delta\phi}$

χ is a kinetic coefficient that determines the crack speed V; we vary the load in order to impose $V = 1$



At V = 1 the fracture energy is well fitted by a two-fold function

$\Gamma(\theta)/\sqrt{\kappa\mathcal{E}_c} = c_1(\epsilon)\cos 2(\theta-\alpha) + c_0(\epsilon)$

ϵ	c_1	c_0	R_Γ
0.33	0.224	2.77	1.176
0.50	0.325	2.84	1.258
1.00	0.573	2.99	1.474
2.00	0.962	3.33	1.812

$R_\Gamma = \Gamma_{max}/\Gamma_m$

Linear Elastic Fracture Mechanic Analysis (1)

Off-center stable crack propagation

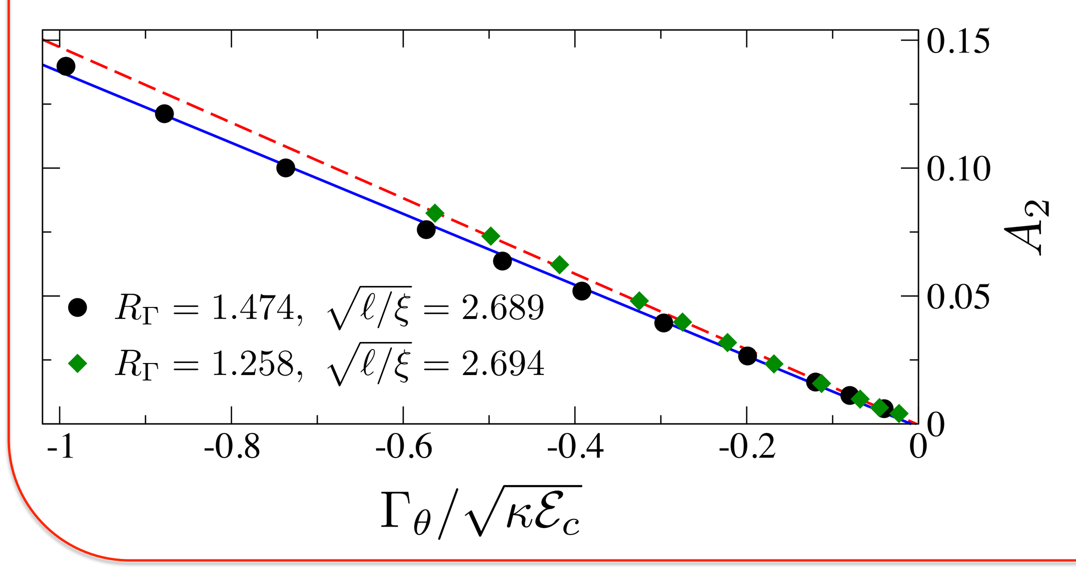
Perturbative expansion of the stress field near the crack tip (angular component)

$$\sigma_{3\theta} = \frac{K_3}{\sqrt{2\pi r}}\cos\frac{\theta}{2} - \mu A_2\sin\theta + \dots \qquad \text{with } K_3 = \mu\Delta\sqrt{2/W}$$

the subdominant term breaks the symmetry of $\sigma_{3\theta}$ around $\theta = 0$

From this expansion and the torque balance condition we obtain the law of motion for the crack tip: $\Gamma_\theta = G_\theta = -K_3A_2\sqrt{\ell}$ → length scale comparable to ξ

For straight cracks $A_2 = y_0\Delta/3W^2\sqrt{1-(y_0/W)^2}$ so that we can measure ℓ from the equilibrium position y_0 for different angles θ



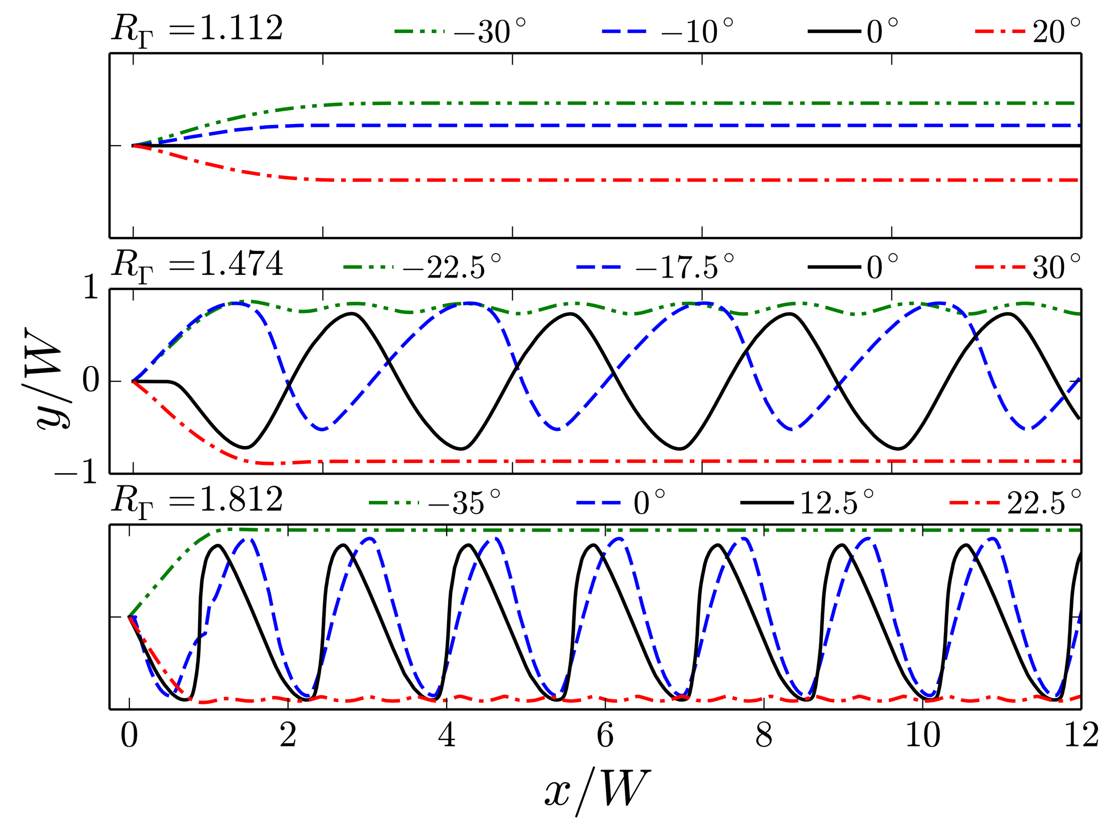
Finally, we are able to predict the exact off-center position of the straight crack:

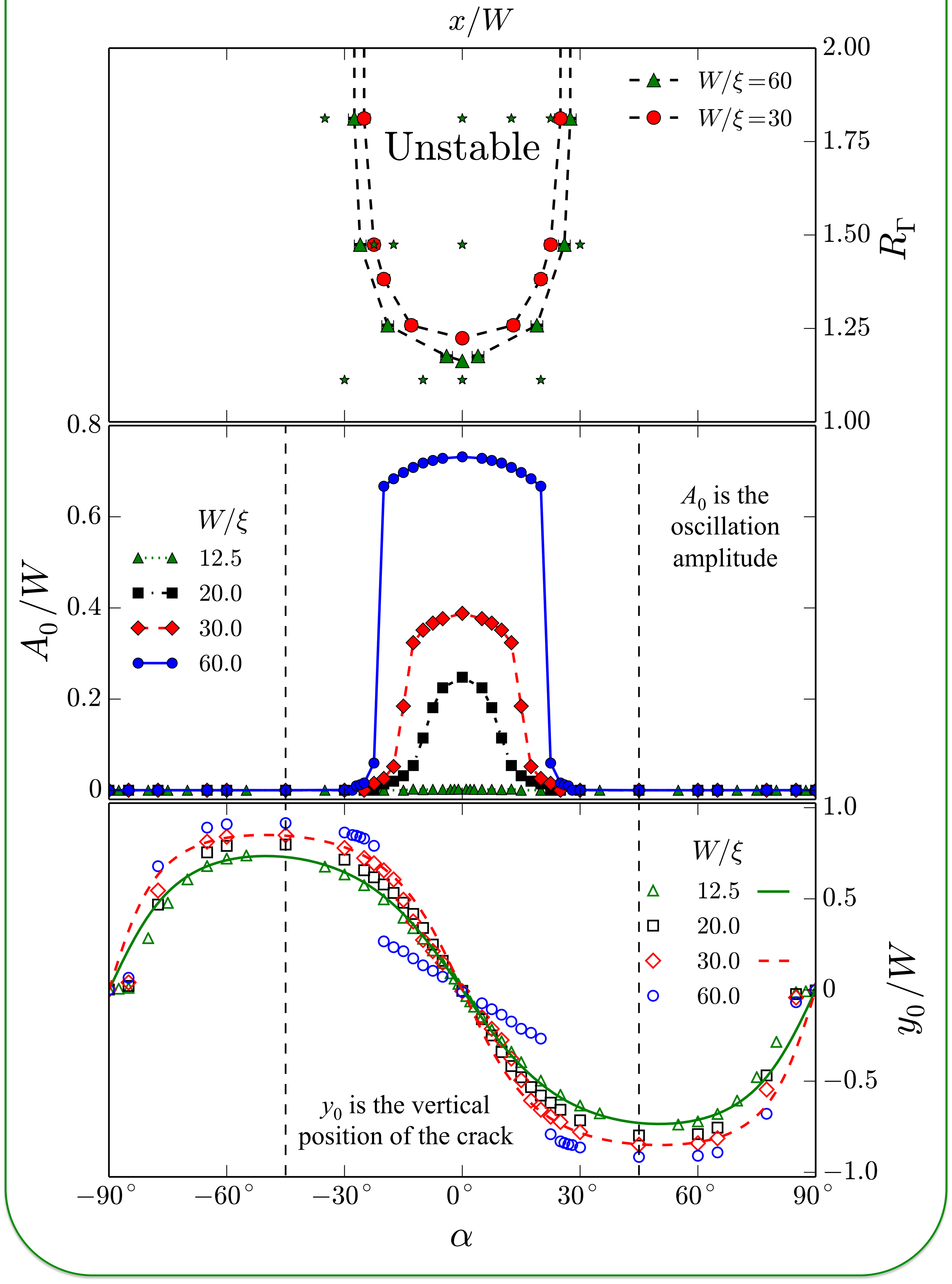
$$\frac{y_0}{W} = -\frac{f_\epsilon}{\sqrt{1+f_\epsilon^2}}$$

where

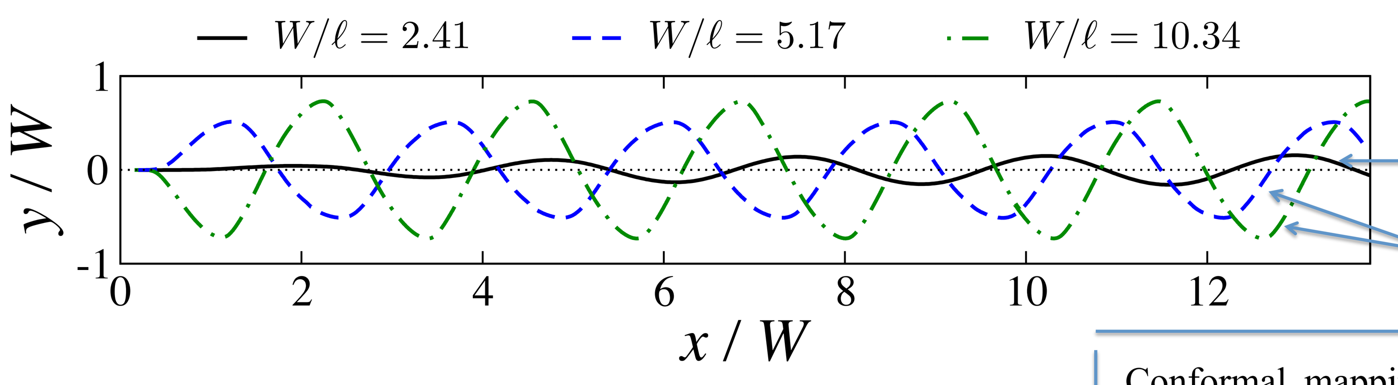
$$f_\epsilon = 3\frac{\Gamma_\theta(0)}{\Gamma(0)}\sqrt{\frac{W}{2\ell}}$$

Phase-field simulations reproduce stable and oscillatory cracks





Linear Elastic Fracture Mechanic Analysis (2) – Oscillatory crack propagation ($\alpha = 0$)

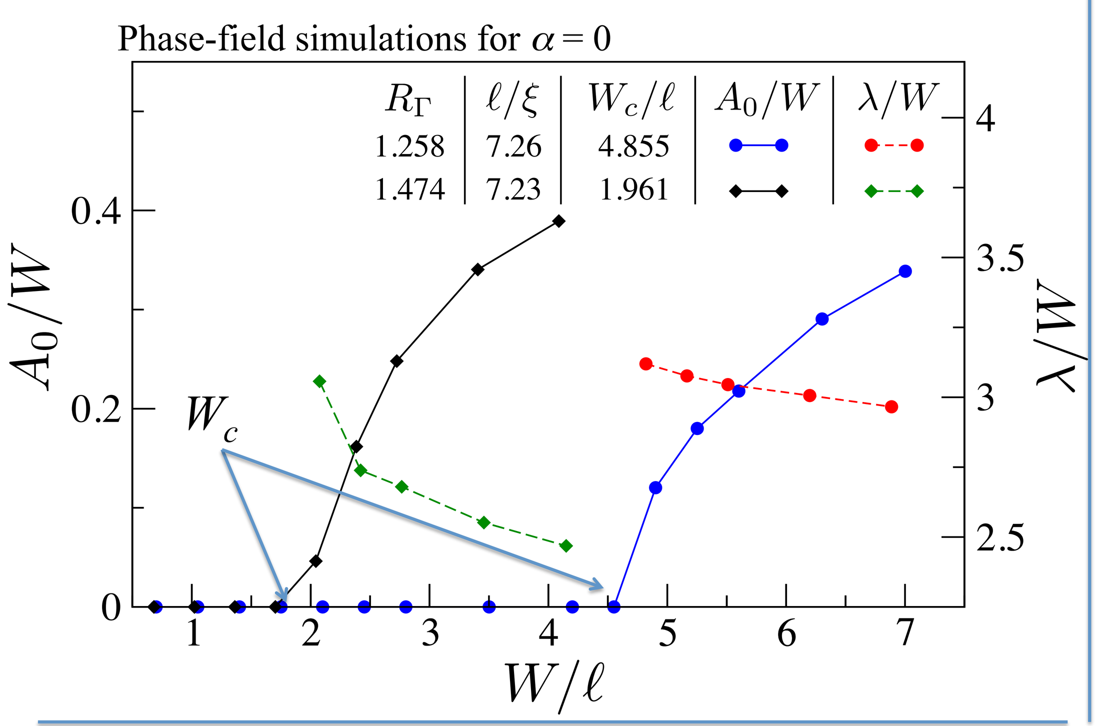


Stabilization of oscillatory cracks due to finite size effects. W is used as bifurcation parameter:

close to the bifurcation threshold ($W = 1.2W_c$)

W largely above W_c (nonlinear paths)

Phase-field simulations for $\alpha = 0$



R_Γ	ℓ/ξ	W_c/ℓ	A_0/W	λ/W
1.258	7.26	4.855		
1.474	7.23	1.961		

Conformal mapping allows one to derive an exact integro-differential equation that provides A_2 at the tip of gently curved cracks

$$A_2 = \frac{\Delta}{W^2}\left[\frac{y_t}{3} + \frac{2W}{\pi}\frac{dy_t}{dx} - \frac{I_1}{2} + I_2\right]$$

where the crack tip is located at (x_t, y_t), while I_1 and I_2 are path dependent integrals:

$I_1 = \int_{\eta_0}^1 d\eta\,[y(\eta)-y(0)]/\eta^{3/2}$

$I_2 = \int_{\eta_0}^1 d\eta\,[y(\eta)-y(0)-\eta\,y'(0)]/\eta^{5/2}$

with $\eta = 1 - e^{-\pi(x_t-x)/W}$ and η_0 is a small scale cutoff of size ξ

The torque balance condition along the direction of maximum fracture energy is $\Gamma_{\theta\theta}\frac{dy_t}{dx} = -K_3A_2\sqrt{\ell}$

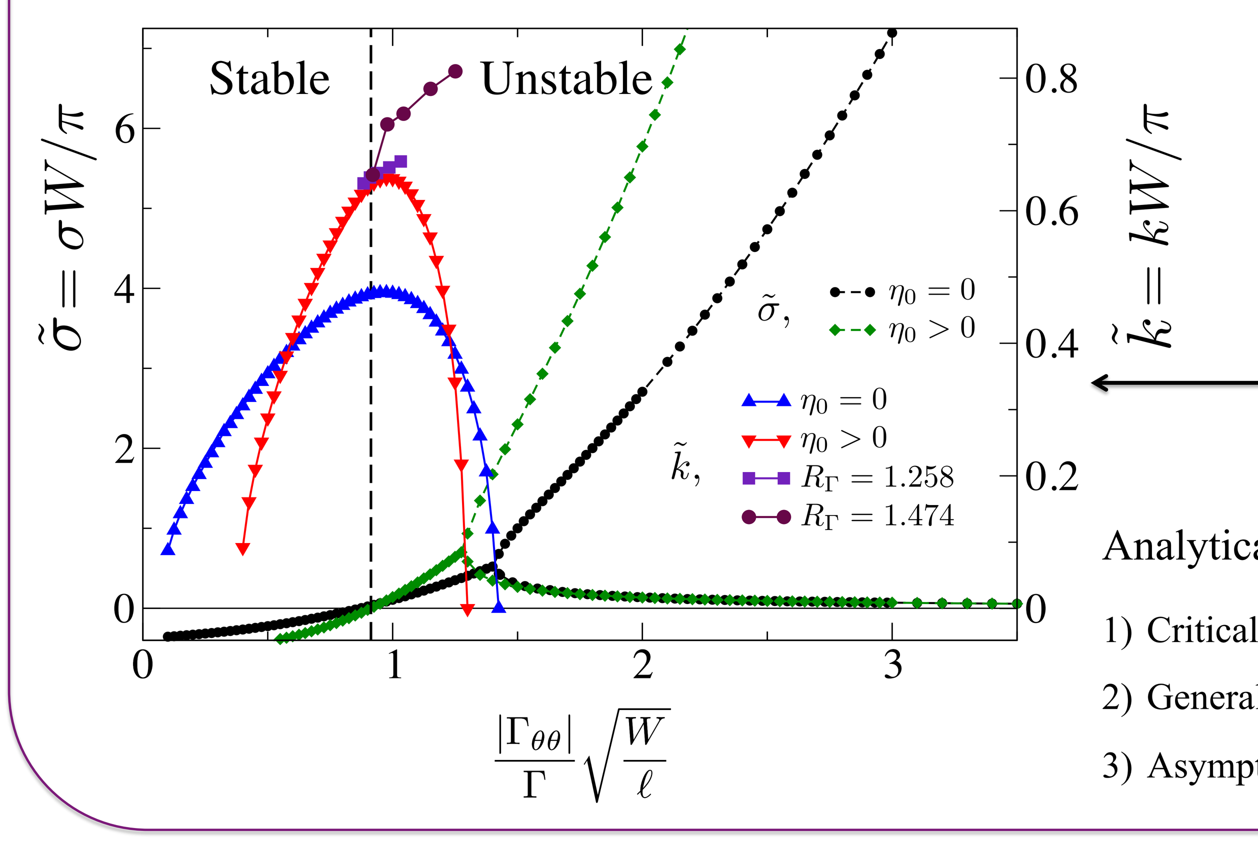
Linear Stability Analysis

$y(x) = \delta_y\,\mathrm{Re}[\exp(\sigma x + ikx)]$

we solve numerically the real and imaginary part of

$$\int_{\eta_0}^1 \frac{d\eta}{\eta^{5/2}}\left[(1-\eta)^{\tilde\sigma}e^{i\tilde k\ln(1-\eta)} - 1 + (\tilde\sigma + i\tilde k)\eta\right] - \frac{1}{2}\int_{\eta_0}^1 \frac{d\eta}{\eta^{3/2}}\left[(1-\eta)^{\tilde\sigma}e^{i\tilde k\ln(1-\eta)} - 1\right] + \frac{1}{3} + \mathcal{C}(\tilde\sigma + i\tilde k) = 0$$

where $\mathcal{C} = 2 + (\pi\Gamma_{\theta\theta}\sqrt{W}/\Gamma\sqrt{2\ell})$



Analytical results for oscillatory cracks:

1) Critical strip width $W_c = c(\eta_0)\,\ell\,(\Gamma/\Gamma_{\theta\theta})^2$

2) General condition for instability $\Gamma_{\theta\theta}/\Gamma < -\sqrt{c(\eta_0)\ell/W}$

3) Asymptotic exp. in the large σ (i.e. size) limit $\sigma \sim \Gamma_{\theta\theta}^2/\Gamma^2\ell$

Conclusions

- The anisotropy of the material induces oscillating fractures when the crack is forced to propagate along directions with $\Gamma_{c\theta\theta} < 0$.
- This condition is necessary but not sufficient because it neglects finite size stabilization effects.
- LEFM analysis proves that oscillatory paths originate from a linear instability of the straight propagating crack.
- Phase-field simulations also reproduce qualitatively the crack paths observed in tearing experiments of anisotropic thin sheets, such as off-center fractures parallel to the strip boundaries and sawtooth patterns.

References:

[1] V. Hakim and A. Karma, Phys. Rev. Lett. **95**, 235501 (2005).
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[3] A. Takei *et al.*, Phys. Rev. Lett. **110**, 144301(2013).
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