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Smooth Particle Hydrodynamic (SPH)

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EGEE 520

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OUTLINE

- Introduction and Historical Perspective:
- General Principles:
- Governing Equations:
- Hand-Calculation Example:
- Comparing Hand Calculation with Results Obtained by the Developed Code:
- Numerical Example and Example Applications:

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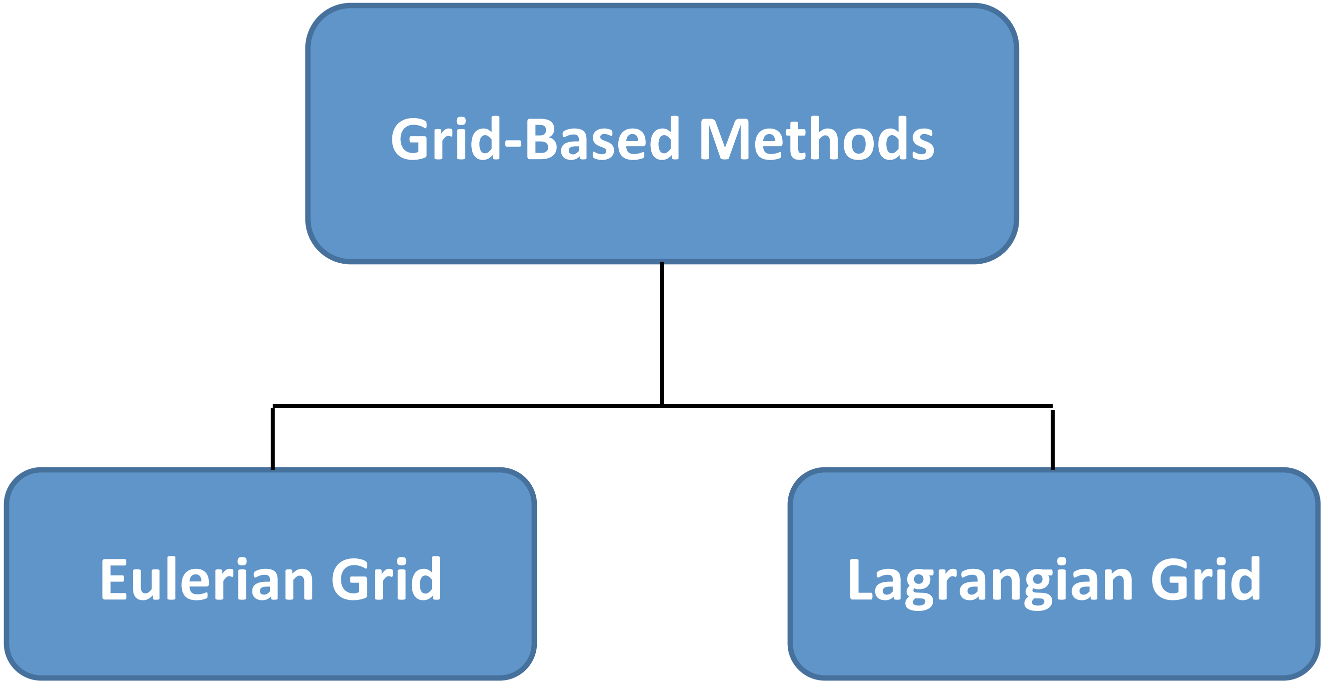
Discretization Methods in Numerical Simulation

Grid-Based Methods

Mesh Free Methods

Discretization Methods in Numerical Simulation



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Limitations of Grid Based Methods

- Eulerian grid methods:
  - Constructing regular grids for irregular geometry

- Lagrangian method;
  - Computing the mesh for the object
  - Large deformation ➔ rezoning techniques

Limitations of Grid Based Methods

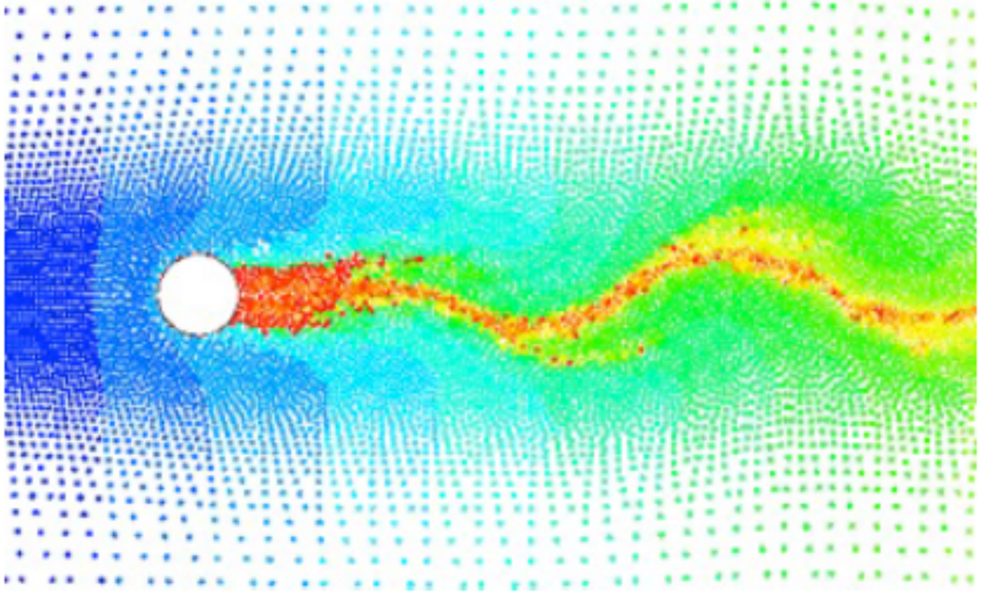
- Not Suitable for problems involvin
  - Large displaceme
  - La omoge
  - Movin
  - Fr
- Hydrody
  - Explo
  - High Velocity act (H

Strong Interest in equivalent Mesh Free Methods

Mesh Free methods

- Accurate and stable numerical solutions for integral equations or PDEs with all kind of boundary conditions

- A set of arbitrary distributed particles without any connectivity between them.



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Mesh Free Particle Methods (MPM)

- Each particle:
  - Directly associated with physical object
  - Represents part of the continuum problem domain
- Particle size:
  - From nano- to micro- to meso- to macro- to astronomical scales.
- The particles posses a set of field variables:
  - Velocity, momentum, energy, position, etc.
- Evolution of the system depends on conservation of:
  - Mass
  - Momentum
  - Energy

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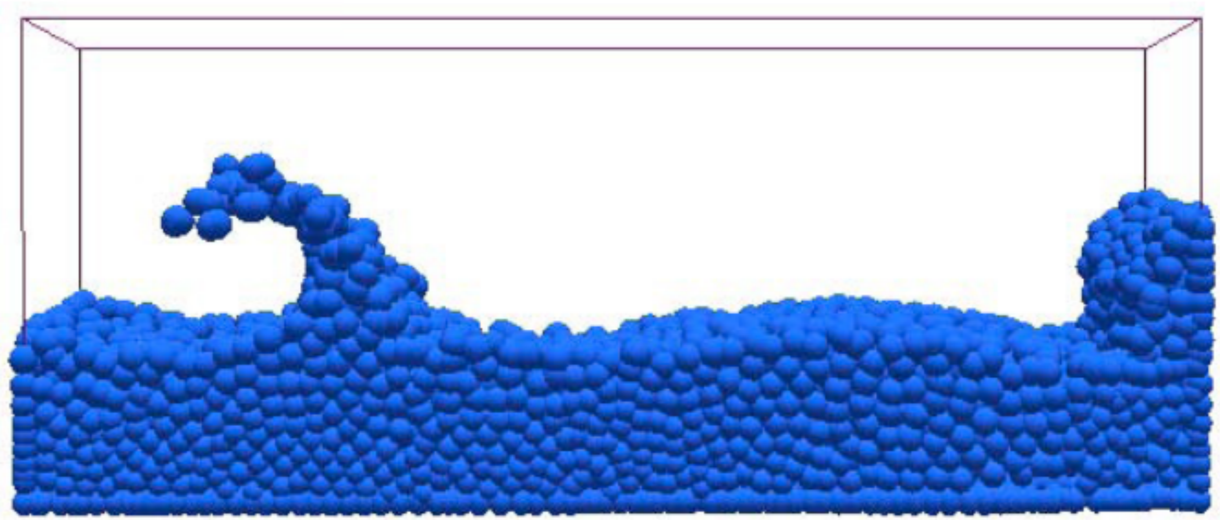
Mesh Free Particle Methods (MPM)

- Inherently lagrangian methods
  - The particles represent the physical system move in the lagrangian framework according to internal interaction and external forces



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Mesh Free Particle Methods (MPM)

✓ Advantages:

- Discretized with particles with no fixed connectivity

  ➔ Good for large deformations

- Simple discretization of complex geometry

- Tracing the motion of the particles

  ➔ Easy to obtain large scale features

- Available time history of all particles

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Smoothed Particle Hydrodynamics (SPH)

- One of the earliest developed mesh free particles methods
  - A mesh free particle method
  - Lagrangian
  - Easily adjustable resolution of the method with respect to variable such as density

- Developed by:
  - Gingold and Monaghan (1977)
  - Lucy (1977)

→ 3D astrophysical problems modeled by classical Newtonian hydrodynamics

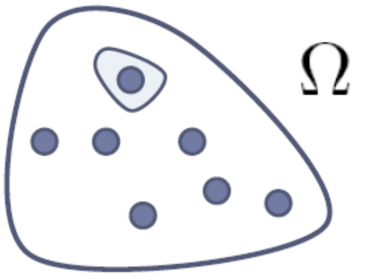
Smoothed Particle Hydrodynamics (SPH)

- Extension:

Fluid Mechanics	B. Solenthaler, 2009	Incompressibility constraints
	Kyle &Terrell, 2013	Full-Film Lubrication
Solid Mechanics	Libersky & Petschek, 1990	Strength of Material problem
	Johnson & Beissel, 1996 Randles & Libersky, 2000	Impact phenomena
	Bonet & Kulasegaram, 2000	Metal forming simulations
	Herreros & Mabssout , 2011	Shock wave propagation in solids

Basic Idea of SPH

- Domain discretization
  - Set of arbitrarily distributed particles
  - No connectivity is needed



→ Smoothing length: spatial distance over which the properties are "smoothed" by a kernel function

- Numerical discretization (at each time step)

  Approximation of functions, derivatives and integrals in the governing equations
  - Particles rather than over a mesh
  - Using the information from neighboring particles in an area of influence

Basic Idea of SPH

- Size of the smoothing length

  - Fixed in space and time

  - Each particle has its own smoothing length varying with time

➔ Automatically adapting the resolution of the solution depending on local condition

➔ Optimising the computational efforts for the regions of interest

✓ Very dense region → many particles are close together

→ relatively short smoothing length

✓ Low-density regions → individual particles are far apart

→ longer smoothing length

Improvement and Modifications

- Issues and limitations associated to SPH:
  - Tensile instability
  - Zero-energy mode

Improvement and Modifications

- Tensile instability

Regions with tensile stress state:

a small perturbation on the positions of particles → particle clumping and oscillatory motion

- Morris (1996) → special smoothing functions
- Dyka(1997) → additional stress points
- Monaghan (2000) → artificial force

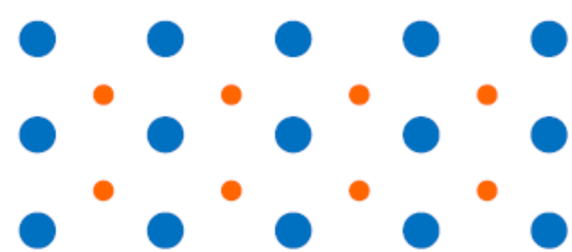
→ Tensile instability remains one of the most critical problems of the SPH method

Improvement and Modifications

- Zero Energy Mode

- Calculating field variables and their derivatives at the same points ➔ Zero gradient of an alternating field variable at the particles

- Also appear in FDM and FEM
- Using 2 types of particles for discretization
  - Velocity particles
  - Stress particles



General Principles

Smoothed Particle Hydrodynamics (SPH)

- Interpolation method → Approximate values and derivatives of continuous field quantities by using discrete sample points.

- The sample points: Smoothed particles that carry:

  1) Concrete entities, e.g. mass, position, velocity

  2) Estimated physical field quantities dependent of the problem, e.g. mass-density, temperature, pressure, etc.

Smoothed Particle Hydrodynamics (SPH)

❖ The basic step of the method

(domain discretization, field function approximation and numerical solution):

- The continuum:

  A set of arbitrarily distributed particles with no connectivity (meshfree);

- Field function approximation:

  The integral representation method

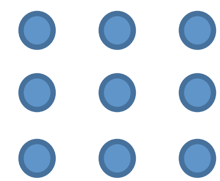
- Converting integral representation into finite summation:

  Particle approximation

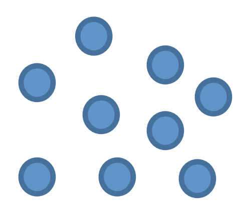
SPH vs Finite Difference Method

- The SPH quantities: Macroscopic and obtained as weighted averages from the adjacent particles.

- Finite difference method : Requires the particles to be aligned on a regular grid



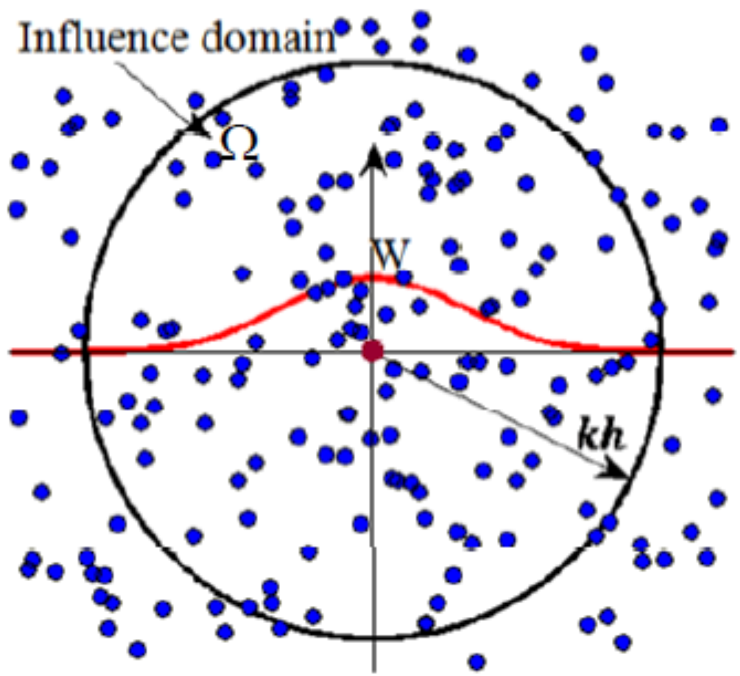
- SPH: Can approximate the derivatives of continuous fields using analytical differentiation on particles located completely arbitrary



Integral representation of a function:

The continuum → A set of arbitrarily particles

$$f(x)=\int_{\Omega} f(x\ )W(x-x\ ,h)dx$$



Integral representation of a function:

- The interpolation is based on the theory of integral interpolants using kernels that approximate a delta function

- The integral interpolant of any quantity function, A(r)

$$A_I(r)=\int_{\Omega} A(r)W(r-r,h)dr$$

- where: r is any point in domain (Ω), W is a smoothing kernel with h as width.

- The width, or core radius, is a scaling factor that controls the smoothness or roughness of the kernel.

➢ Integral representation into finite summation

➢ Numerical equivalent

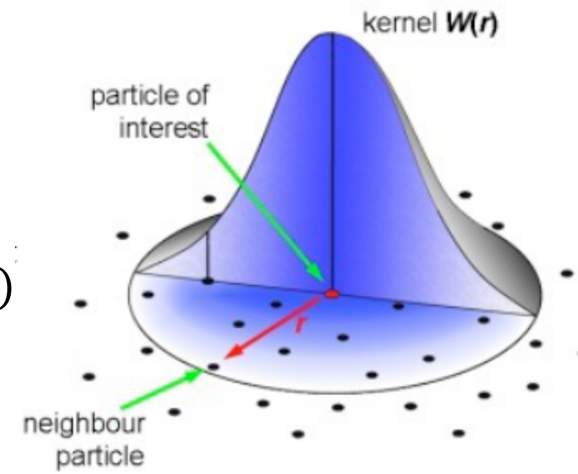
$$A_I(r)=\int_\Omega A(r')W(r-r',h)dr'$$

$$\rightarrow A_S(r)=\sum_j A_j V_j W(r-r_j,h)$$

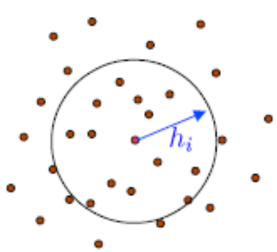
➢ where j is iterated over all particles, V_j is the volume attributed implicitly to particle j, r_j the position, and A_j is the value of any quantity A at r_j

$$V=m/\rho$$

Smoothing function in support domain



➢ The basis formulation of the SPH



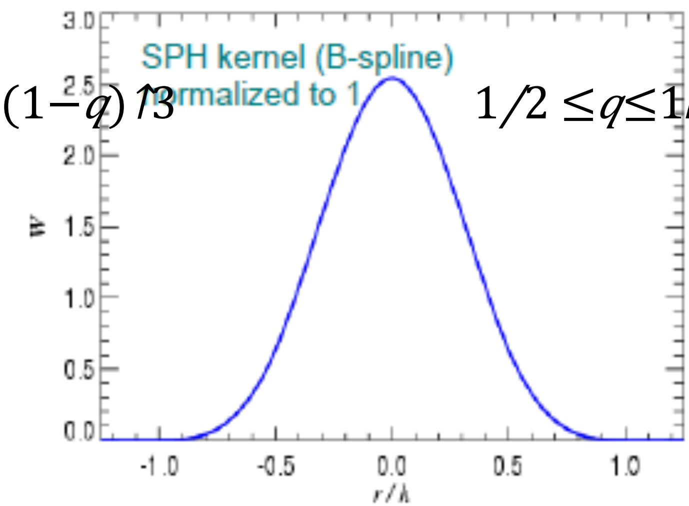
$$A_S(r)=\sum_j A_j \frac{m_j}{\rho_j} W(r-r_j,h)$$

Smoothing Kernels

- Must be normalized to unity
- High order of interpolation
- Spherical symmetry (for angular momentum conservation)

$$W(q)=\frac{8}{\pi}\begin{cases}1-6q^2+6q^3 & 0\le q\le 1/2\\ 2(1-q)^3 & 1/2\le q\le 1\end{cases}$$

$$q=r_{j,i}/h$$



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Kernel Function

- **Kernel function used in hand calculation and in the code**

$$W(r,h)=\frac{1}{\pi * h^{3}} * \begin{cases} 1+\frac{3}{4} * q^{3}+\frac{3}{2} * q^{2} & if\ 0\leq q\leq 1 \\ \frac{1}{4} * (2-q)^{3} & if\ 1\leq q\leq 2 \\ 0 & otherwise \end{cases}$$

where:

$$q=r_{j,i}/h$$

Smoothing Kernels

- **Properties of Kernel**

Normalization condition $\int_{\Omega} W(r,h)\,dr = 1$

$$\lim_{h \to 0} W(r,h) = \delta(r)$$

where: $\delta(r) = \begin{cases} \infty & \|r\| = 0 \\ 0 & oth \end{cases}$

Must also be positive $W(r,h) \geq 0$

Even function $W(r,h) = W(-r,h)$

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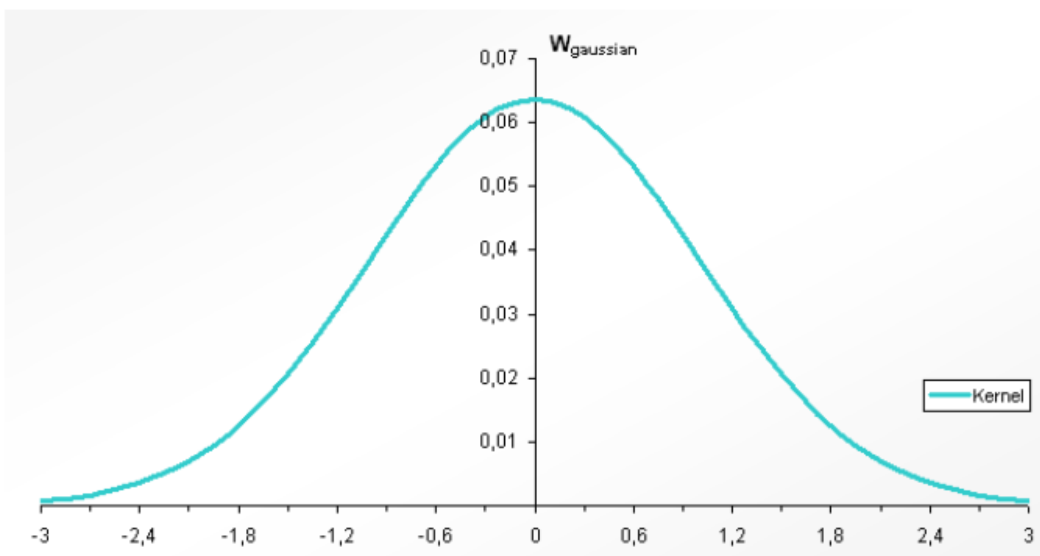
Smoothing Kernels

- The first golden rule: If you want to find a physical interpretation then it is always best to assume the kernel is Gaussian

$$W_{gaussian}(r,h)=\frac{1}{(2\pi h^{2})^{3/2}}e^{-\left(\frac{\|r\|^{2}}{2h^{2}}\right)}, \quad h>0$$



The isotropic Gaussian kernel in 1D, for h=1

The Gradient and the Laplacian of a quantity field

$$\frac{\partial}{\partial x} A_S(r) = \frac{\partial}{\partial x} \sum_j \left(A_j \frac{m_j}{\rho_j} W(r - r_j, h) \right)$$

- Using the product rule

$$\frac{\partial}{\partial x} \left(A_j \frac{m_j}{\rho_j} W(r - r_j, h) \right) = \frac{\partial}{\partial x} \left(A_j \frac{m_j}{\rho_j} \right) W(r - r_j, h) + A_j \frac{m_j}{\rho_j} \frac{\partial}{\partial x} W(r - r_j, h)$$

$$= 0\, W(r - r_j, h) + A_j \frac{m_j}{\rho_j} \frac{\partial}{\partial x} W(r - r_j, h)$$

$$= A_j \frac{m_j}{\rho_j} \frac{\partial}{\partial x} W(r - r_j, h)$$

The Gradient and the Laplacian of a quantity field

$$\nabla A_S(r) = \sum_j A_j \frac{m_j}{\rho_j} \nabla W(r - r_j, h)$$

- To obtain higher accuracy on the gradient of a quantity field the interpolant can instead be obtained by using

$$\nabla(\rho A) = \rho \nabla A + A \nabla \rho \quad \longleftrightarrow \quad \rho \nabla A = \nabla(\rho A) - A \nabla \rho \qquad (*)$$

$$\nabla A = 1/\rho \, (\nabla(\rho A) - A \nabla \rho)$$

- The second golden rule: Rewrite formulas with density inside operators

The Gradient and the Laplacian of a quantity field

$$\nabla A_S(r) = \frac{1}{\rho}\left[\sum_j \rho_j A_j \frac{m_j}{\rho_j} \nabla W(r-r_j,h) - A\sum_j \rho_j \frac{m_j}{\rho_j} \nabla W(r-\right.$$

$$= \frac{1}{\rho}\left[\sum_j A_j m_j \nabla W(r-r_j,h) - \sum_j A m_j \nabla W(r-r_j,h)\right]$$

$$= \frac{1}{\rho}\sum_j (A_j - A) m_j \nabla W(r-r_j,h)$$

- A particular symmetrized form of (*) can be obtained be rewriting

$$\nabla\left(\frac{A}{\rho}\right) = \frac{\nabla A}{\rho} - \frac{A}{\rho^2}\nabla\rho \quad \longleftrightarrow \quad \frac{\nabla A}{\rho} = \nabla\left(\frac{A}{\rho}\right) + \frac{A}{\rho^2}\nabla\rho$$

$$\nabla A = \rho\left(\nabla\left(\frac{A}{\rho}\right) + \frac{A}{\rho^2}\nabla\rho\right)$$

What we have: $\nabla A_S(r) = \sum_j A_j \frac{m_j}{\rho_j} \nabla W(r-r_j,h)$ $\quad \nabla A = \frac{1}{\rho}\left(\nabla(\rho A) - A\nabla\rho\right)$

The Gradient and the Laplacian of a quantity field

- Which in SPH terms becomes

$$\nabla A_S(r) = \rho\left[\sum_j \frac{A_j}{\rho_j}\frac{m_j}{\rho_j}\nabla W(r-r_j,h) + \frac{A}{\rho^2}\sum_j \rho_j \frac{m_j}{\rho_j}\right.$$

$$= \rho\left[\sum_j \frac{A_j}{\rho_j}\frac{m_j}{\rho_j}\nabla W(r-r_j,h) + \sum_j \frac{A}{\rho^2} m_j \nabla W(r-r_j,h)\right]$$

$$= \rho\sum_j \left(\frac{A_j}{\rho_j^2} + \frac{A}{\rho^2}\right) m_j \nabla W(r-r_j,h)$$

- The Laplacian of the smoothed quantity field

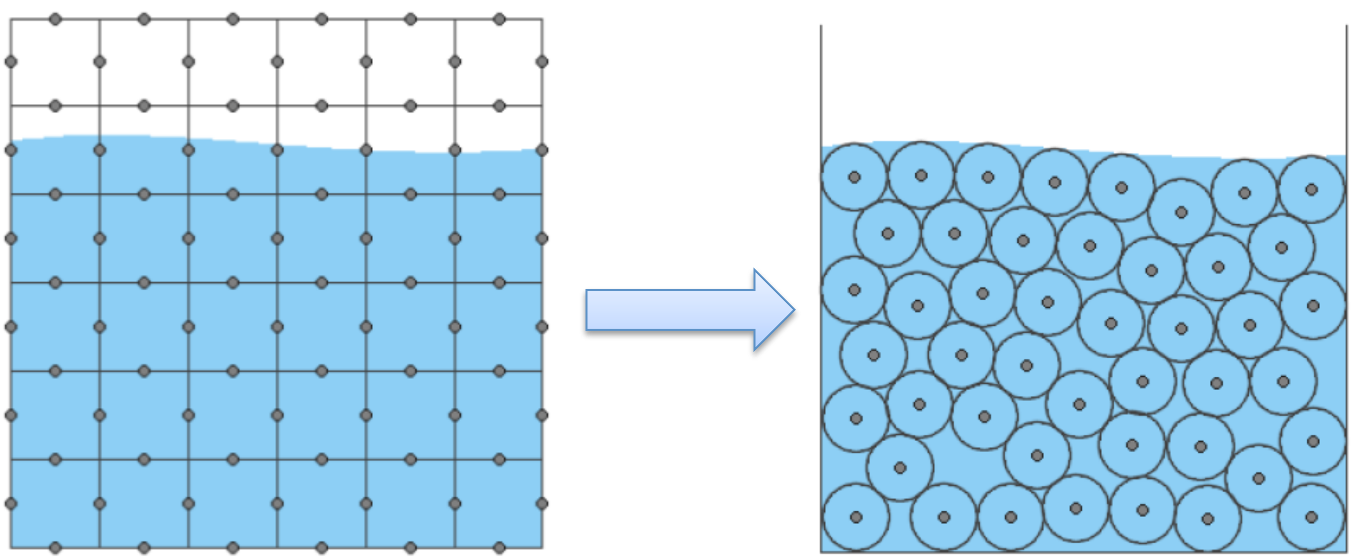
$$\nabla^2 A_S(r) = \sum_j A_j \frac{m_j}{\rho_j}\nabla^2 W(r-r_j,h)$$

What we have: $\nabla A_S(r) = \sum_j A_j \frac{m_j}{\rho_j}\nabla W(r-r_j,h)$ $\nabla A = \rho\left(\nabla\left(\frac{A}{\rho}\right) + \frac{A}{\rho^2}\nabla\rho\right)$

Solving Navier-Stokes by SPH

- Navier-Stokes equations for an incompressible, isothermal fluid

$$\rho du/dt = -\nabla p + \mu\tau\nabla\uparrow 2\ u + f$$



The Basis Formulations of SPH- Summary

$$A_S(r)=\sum_j A_j \frac{m_j}{\rho_j} W(r-r_j,h)$$

$$\nabla A_S(r)=\sum_j A_j \frac{m_j}{\rho_j} \nabla W(r-r_j,h)$$

$$\nabla^2 A_S(r)=\sum_j A_j \frac{m_j}{\rho_j} \nabla^2 W(r-r_j,h)$$

$$\langle f_1+f_2\rangle=\langle f_1\rangle+\langle f_2\rangle$$

$$\langle f_1 f_2\rangle=\langle f_1\rangle\langle f_2\rangle$$

$$\langle c f_2\rangle=c\langle f_2\rangle$$

- A symmetrized gradient of a higher accuracy can in SPH be obtained by

$$\nabla A_S(r)=\rho\sum_j \left(\frac{A_j}{\rho_j^2}+\frac{A}{\rho^2}\right) m_j \nabla W(r-r_j,h)$$

Governing Equations

Governing Equations

- Conservation of

Mass

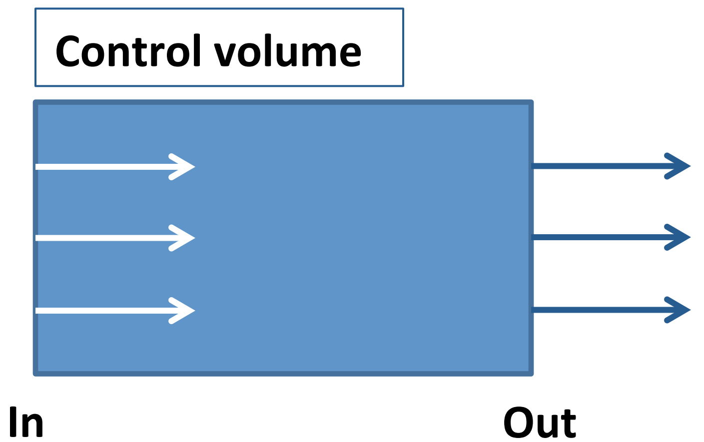
--Diffusion Equation (Fick’s law)

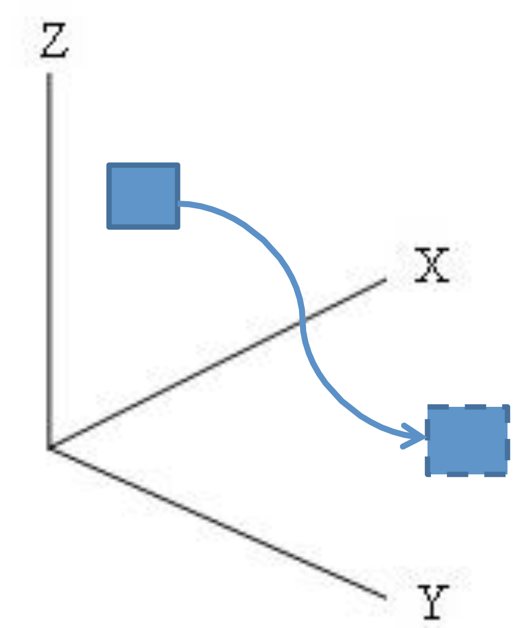
Momentum (Newton second law)

--Navier-Stokes Equation

Energy (first law of thermodynamics)

Eulerian vs. Lagrangian representations

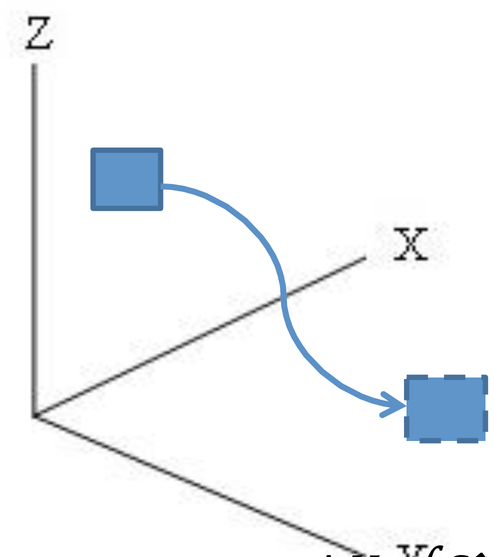


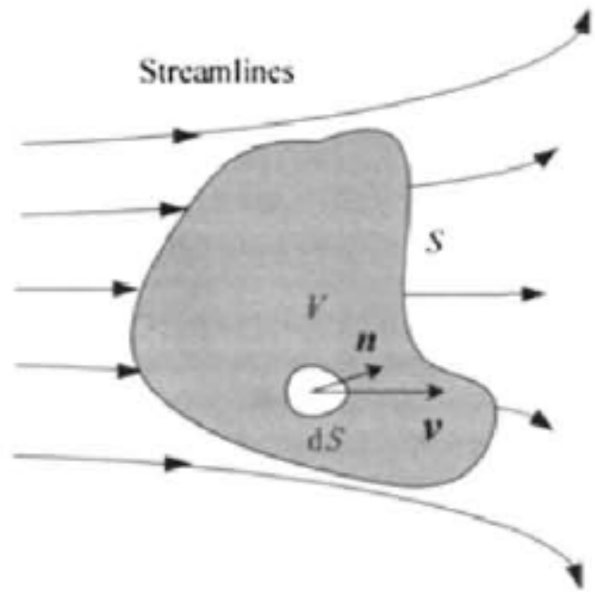


- Space fixed
- Fluid inside control volume changes

- Fluid parcel in material volume
- Carried along with flow

Lagrangian Form



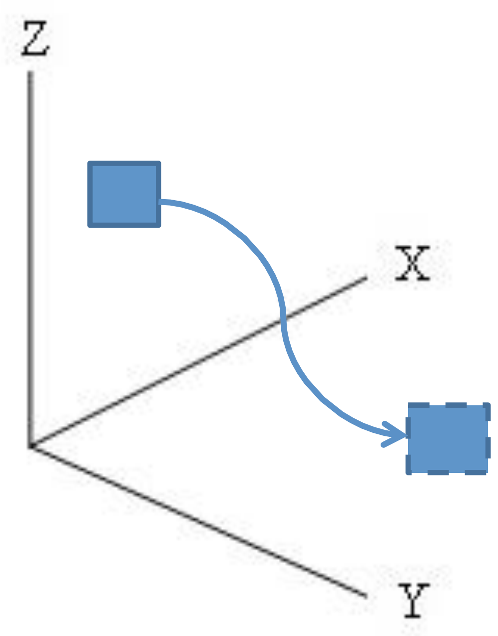


$$\Delta V = \int_S \mathbf{v}\,\Delta t \cdot \boldsymbol{n}\, dS$$

$$\Delta(\delta V)/\Delta t = (\nabla \cdot \mathbf{v}) \int_S d(\delta V) = \nabla \cdot \mathbf{v}\, \delta V$$

$$\nabla \cdot \mathbf{v} = 1/\delta V \; d(\delta V)/dt$$

Continuity Eq.



Mass conserved in a Lagrangian fluid cell

Conservation of mass

$\delta m = \rho \delta V$

$d(\delta m)/dt = d(\rho \delta V)/dt = \delta V d\rho/dt + \rho\, d(\delta V)/dt = 0$

$d\rho/dt = -\rho/\ \delta V\ d(\delta V)/dt = -\rho \nabla \cdot \mathbf{v}$

$d\rho/dt = -\rho \nabla \cdot \mathbf{v}$

Continuity Eq. in SPH

Continuity Eq.

$$\frac{d\rho}{dt} = -\rho \nabla\cdot\mathbf{v}$$

$$= -\rho\left[\nabla\cdot(\rho\mathbf{v}) - \mathbf{v}\cdot\nabla\rho/\rho\right]$$

Gradient Approximation in SPH:

$$\nabla A_S(r) = \frac{1}{\rho}\sum_j (A_j - A)\, m_j \nabla \boldsymbol{W}(r - r_j, h)$$

$$\nabla_i A_i = \frac{1}{\rho_i}\sum_{j=1}^{N} m_j (A_j - A_i)\, \nabla_i \boldsymbol{W}_{ij}$$

$$\frac{d\rho_i}{dt} = -\sum_{j=1}^{N} m_j (\mathbf{v}_i - \mathbf{v}_j)\cdot\nabla_i \boldsymbol{W}_{ij}$$

Momentum equation in three dimensions

- Surface forces

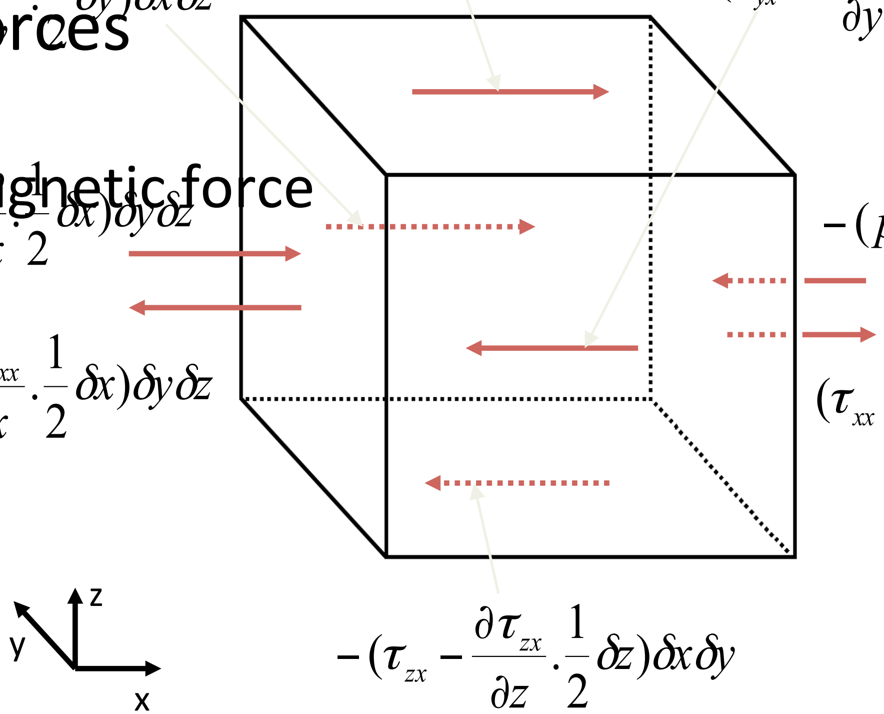
--pressure

--viscous force

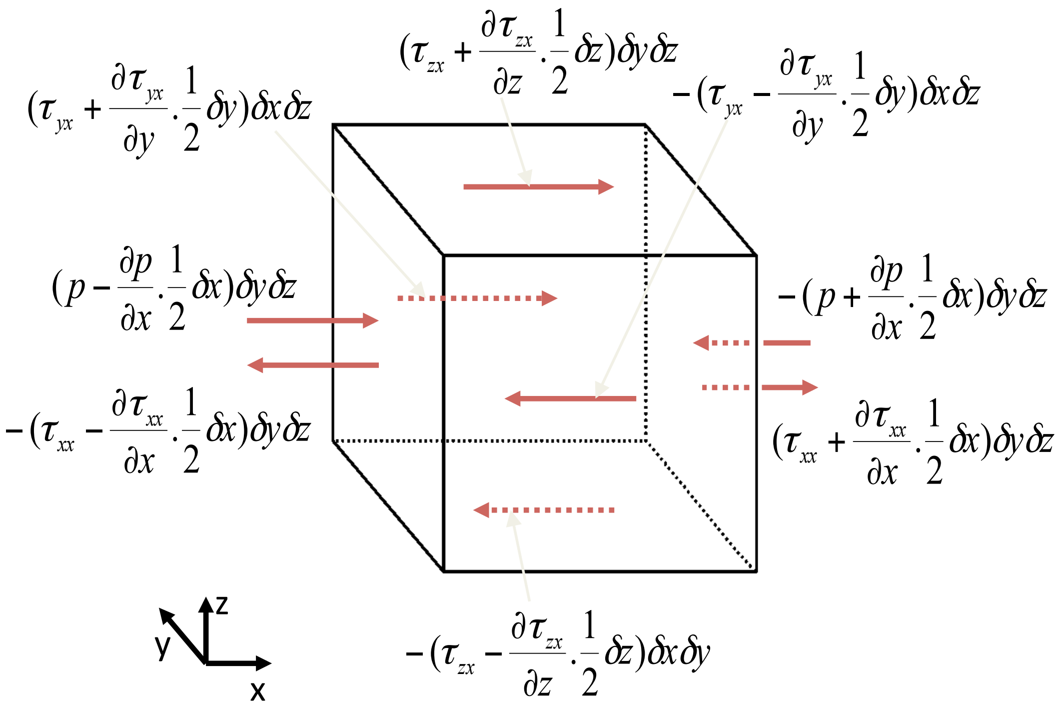
- Body forces

--gravity

--electromagnetic force



Momentum equation



Pressure acting on the fluid cell

$$-[(p+\partial p/\partial x)-p]dydz=-\partial p/\partial x\, dxdydz$$

Stress acting on the fluid cell

$$(\partial \tau\downarrow xx/\partial x+\partial \tau\downarrow yx/\partial y+\partial \tau\downarrow zx/\partial y)dx$$

（x direction)

Momentum equation

Newton's second law

$$m\frac{dv_x}{dt} = \rho\, dxdydz \frac{dv_x}{dt} = -\frac{\partial p}{\partial x} dxdydz + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial\ \ldots}\right.$$

$$\rho \frac{dv_x}{dt} = \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial y}$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \nabla \cdot \boldsymbol{\tau}$$

$$\tau_{ij} = \mu\left(\frac{\partial \mathbf{v}_j}{\partial \boldsymbol{x}_i} + \frac{\partial \mathbf{v}_i}{\partial \boldsymbol{x}_j} - \frac{2}{3}(\nabla \cdot \mathbf{v})\delta_{ij}\right)$$

Momentum Eq. in SPH

Momentum Eq.

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \nabla \cdot \tau$$

Gradient Approximation in SPH:

$$\nabla A_S(\mathbf{r}) = \frac{1}{\rho} \sum_j (A_j - A)\, m_j \nabla \boldsymbol{W}(\mathbf{r} - \mathbf{r}_j, h)$$

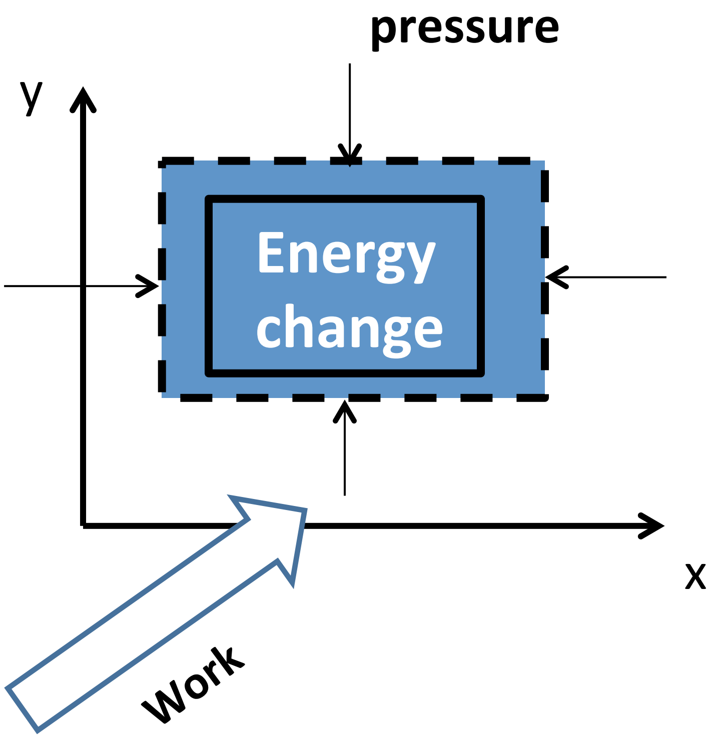
$$\nabla_i A_i = \frac{1}{\rho_i} \sum_{j=1}^{N} m_j (A_j - A_i) \nabla_i \boldsymbol{W}_{ij}$$

$$\frac{d\mathbf{v}_i}{dt} = -\sum_{j=1}^{N} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} + \Pi_{ij} \right) \nabla_i \boldsymbol{W}_{ij}$$

$$\Pi_{ij} = \begin{cases} \dfrac{(\ldots \mu_{ij} + \beta \mu_{ij}^2)}{\rho_{ij}} \\ 0 \end{cases}$$

$$\mu_{ij} = \frac{h(\mathbf{v}_i - \mathbf{v}_j)\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)}{\boldsymbol{r}_{ij}^2 + 0.01h^2}$$

Energy Eq.



Deformation in x direction:

$$(v_x + \partial v_x/\partial x - v_x)dt = \partial v_x/\partial x \, dxdt$$

Work acting on the fluid cell:

$$dydz \partial v_x/\partial x \, dxdt + pdxdz \partial v_y/\partial y \, dydt + pdxdy \partial v_z/\partial z \, dzdt = pdtdxdydz(\partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z)$$

Internal energy change: $\rho de \delta V$

$$\rho de/dt = -p(\partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z /\partial$$

Energy Eq. in SPH

Energy Eq.

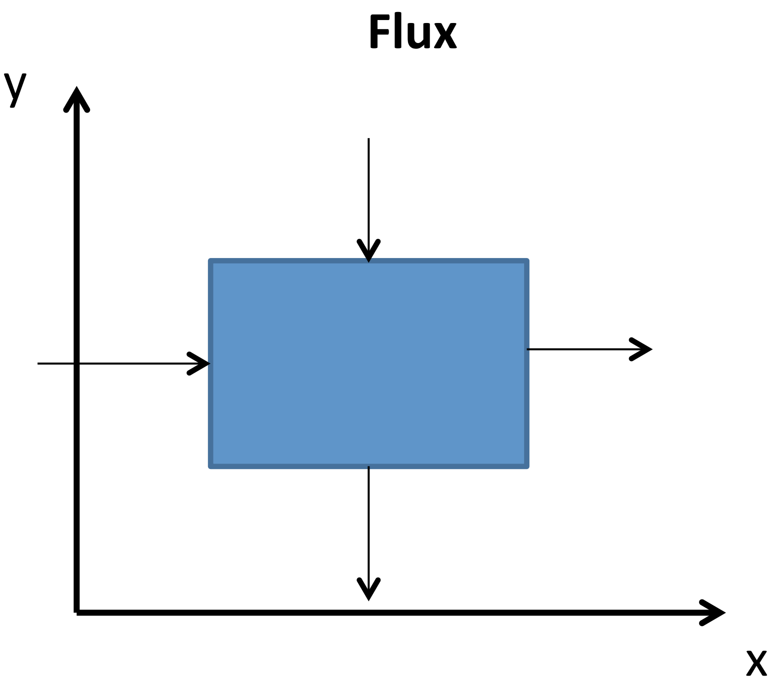
$$\frac{de}{dt} = -\frac{p}{\rho}\frac{\partial \mathbf{v}}{\partial \boldsymbol{x}}$$

Gradient Approximation in SPH:

$$\nabla_i A_i = \frac{1}{\rho_i}\sum_{j=1}^{N} m_j \left(A_j - A_i\right) \nabla_i W$$

$$\frac{de_{\boldsymbol{i}}}{dt} = \frac{p_i}{\rho_i^2}\sum_{j=1}^{N} m_j \left(\mathbf{v}_i - \mathbf{v}_j\right)\nabla_i \boldsymbol{W}_{ij} + \frac{1}{2}\sum_{j=1}^{N} m_j \Pi_{ij} ($$

Diffusion Eq.



Flux change(x direction)

$$u_x \, dx = D[(C + \partial C/\partial x) - C]dydz = D\partial C/\partial x \; dxdydz$$

Fick's law (x direction)

$$u_x = DA_x \, \partial C/\partial x$$

Conservation of mass

$$\delta V dC/dt = u_x \, dx + u_y \, dy + u_z \, dz$$

$$dC/dt = D\nabla^2 C$$

Diffusion Eq. in SPH

Diffusion Eq.

$$\frac{dC}{dt} = D\nabla^2 C = \frac{1}{\rho}\nabla\cdot(D\rho\nabla C)$$

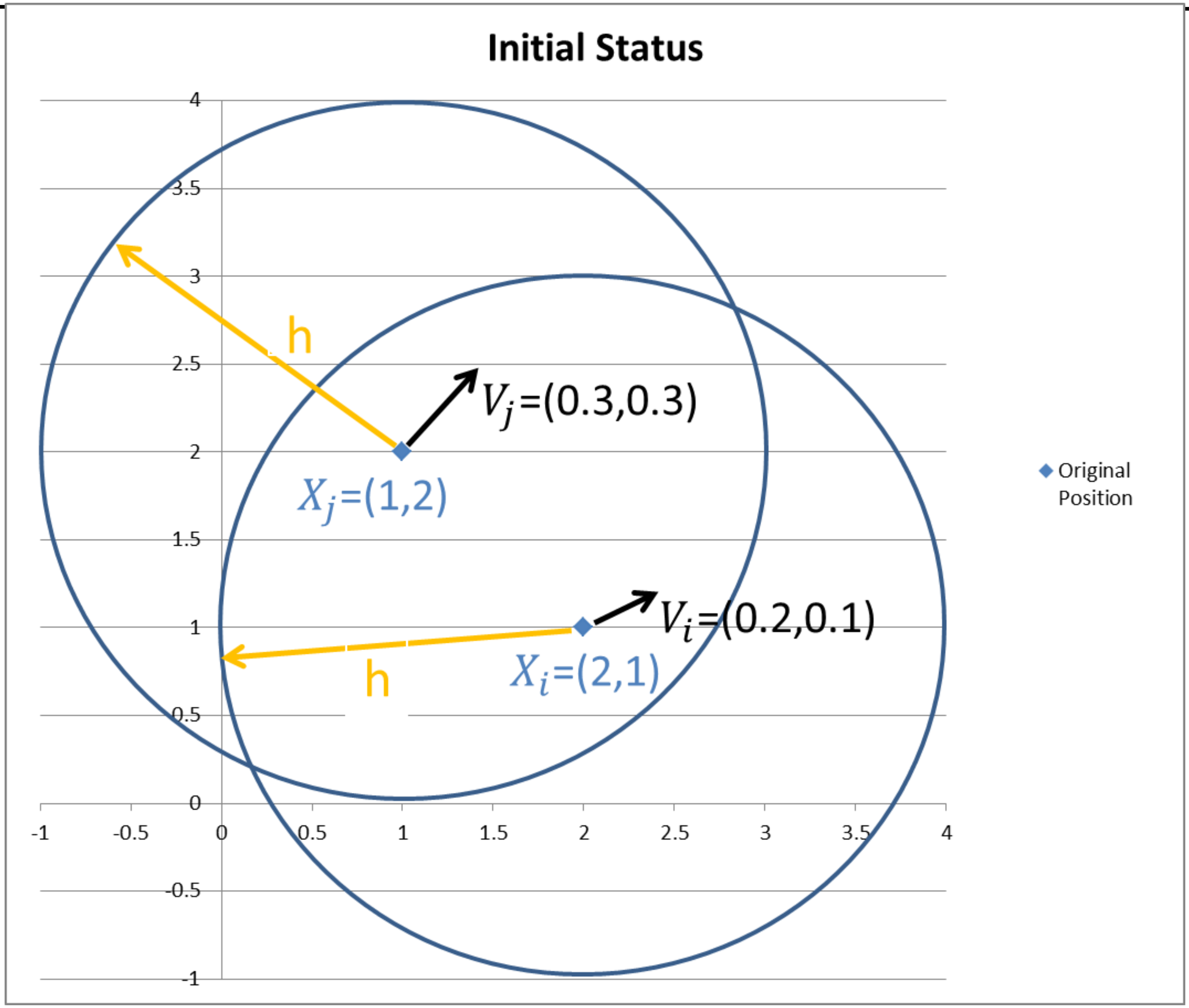
Gradient Approximation in SPH:

$$\nabla_i A_i = \frac{1}{\rho_i}\sum_{j=1}^{N} m_j (A_j - A_i)\nabla$$

$$\frac{dC_i}{dt} = \sum_{j=1}^{N} \frac{m_j}{\rho_i \rho_j} \frac{(D_i + D_j)(\rho_i + \rho_j)\mathbf{r}_{ij}\cdot\nabla_i \mathbf{W}_{ij}}{r_{ij}^2 + \eta^2} (C_i - C_j)$$

Hand-Calculation Example

Problem Description



Initial Parameters

	Mass	Density	Pressure	Velocity	Location	h	Δt
Particle i	1	1	1	(0.2.0.1)	(2,1)	2	1
Particle j	1	1	1	(0.3,0.3)	(1,2)	2	1

Governing Equation

According to the momentum equation,

For Particle i

$$\frac{dV_i}{dt} = -m_j * \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \prod_{i,j}\right) * \nabla W_{i,j} + g$$

No Gravity

$$\frac{dV_i}{dt} = -m_j * \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \prod_{i,j}\right) * \nabla W_{i,j}$$

Calculation of Viscosity Tensor

$$\Pi_{i,j} = \begin{cases} -a_M * 0.5 * (C_{si} + C_{sj}) * \mu_{ij} + \beta * \mu_{ij}^{2} / 0.5 * (\rho_i + \rho_j) & \text{if } V_{ij} \cdot X_{ij} < 0 \\ 0 & \text{if } V_{ij} \cdot X_{ij} > 0 \end{cases}$$

In my case

$$V_{ij} = V_i - V_j = (-0.1, -0.2)$$

$$X_{ij} = X_i - X_j = (1, -1)$$

$$V_{ij} \cdot X_{ij} = -0.1 + 0.2 = 0.1 > 0$$

$$\Pi_{i,j} = 0;$$

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Calculation of $\nabla W_{i,j}$

In my case,

$W(r,h)=\frac{1}{\pi * h^3} * \begin{cases} 1+\frac{3}{4} * q^3 + \frac{3}{2} * q^2 & \text{if } 0 \le q \le 1 \\ \frac{1}{4} * (2-q)^3 & \text{if } 1 \le q \le 2 \\ 0 & \text{otherwise} \end{cases}$

q = r_{ij}/h In this case, $r_{ij} = |X_i - X_j| = \sqrt{2}$,q= $\sqrt{2}/2$

So ~~$W_{i,j} = \frac{1}{\pi * h^3} * (1+\frac{3}{2} * q^2 + \frac{3}{4} * q^3) =$~~

$\frac{1}{\pi * h^3} + \frac{3r^2}{2\pi * h^5} + \frac{3r^3}{4\pi * h^6}$

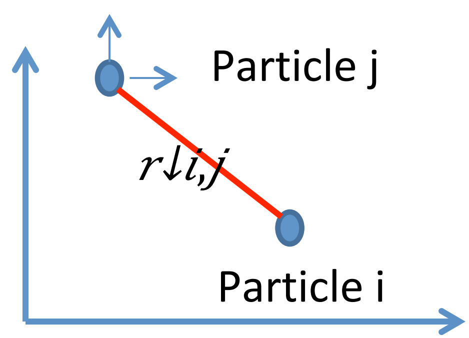
Calculation of $\nabla W_{i,j}$

Using the numerical method to solve $\nabla W_{i,j}$

$$\frac{\partial W_{ij}}{\partial x} = \left(\frac{1}{\pi * h^3} + \frac{3*((1-0.001)^2+1^2)}{2\pi * h^5} + \frac{3*((1-0.001)^2+1^2)^{3/2}}{4\pi * h^6}\right) - \left(\frac{1}{\pi * h^3} + \frac{3*2}{2\pi * h^5} + \frac{3*2^{3/2}}{4\pi * h^6}\right) / 0.001 = -4.57*10^{(-2)}$$

$$\frac{\partial W_{ij}}{\partial y} = \left(\frac{1}{\pi * h^3} + \frac{3*(1^2+(1+0.001)^2)}{2\pi * h^5} + \frac{3*(1^2+(1+0.001)^2)^{3/2}}{4\pi * h^6}\right) - \left(\frac{1}{\pi * h^3} + \frac{3*2}{2\pi * h^5} + \frac{3*2^{3/2}}{4\pi * h^6}\right) / 0.001 = 4.57*10^{(-2)}$$

$\nabla W_{i,j} = \frac{\partial W_{ij}}{\partial x}, \frac{\partial W_{ij}}{\partial y} = (-4.57*10^{-2}, 4.57*10^{-2});$



Calculation of Acceleration, Velocity and Location

$$a_i = \frac{dV_i}{dt} = -m_j * \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \prod_{i,j}\right) * \nabla W_{i,j}$$

$$= -1 * \left(\frac{1}{1^2} + \frac{1}{1^2} + 0\right) * \left(-4.57 * 10^{-2}, 4.57 * 10^{-2}\right)$$

$$= \left(9.14 * 10^{-2}, -9.14 * 10^{-2}\right);$$

$$\mathbf{V_{i\,new}} = \mathbf{V_i} + \mathbf{a_i} * \Delta\mathbf{t} = (\mathbf{0.29}, \mathbf{0.01});$$

$$\mathbf{X_{i\,new}} = \mathbf{X_i} + \mathbf{V_{i\,new}} * \Delta\mathbf{t}$$

=(2.29,1.01);

Calculation of Viscosity tensor for Particle j

$$\frac{dV_j}{dt} = -m_i * \left(\frac{P_j}{\rho_j^2} + \frac{P_i}{\rho_i^2} + \prod_{j,i}\right) * \nabla W_{j,i}$$

← **Governing Equation**

$$\prod_{j,i} = \begin{cases} \dfrac{-a_M * 0.5 * (C_{si} + C_{sj}) * \mu_{ij} + \beta * \mu_{ij}^2}{0.5 * (\rho_i + \rho_j)} & \text{if } V_{ji} \cdot X_{ji} < 0 \\ 0 & \text{if } V_{ji} \cdot X_{ji} > 0 \end{cases}$$

$$V_{j,i} = V_i - V_j = (0.1, 0.2)$$

$$X_{j,i} = X_i - X_j = (-1, 1)$$

$$V_{ji} \cdot X_{ji} = -0.1 + 0.2 = 0.1 > 0$$

$$\prod_{j,i} = 0;$$

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Calculation of $\nabla W_{j,i}$ for Particle j

$$W(r,h) = 1/\pi * h^3 * \{ ■ 1 + 3/4 * q^3 + 3/2 * q^2$$

$$if\ 0 \le q \le 1 \square 1/4 * (2-q)^3 \qquad if$$

$$1 \le q \le 2 \square 0$$

$$otherwise$$

q = $r_{j,i}/h \qquad r_{j,i} = |X_j - X_i| = \sqrt{\square} 2$

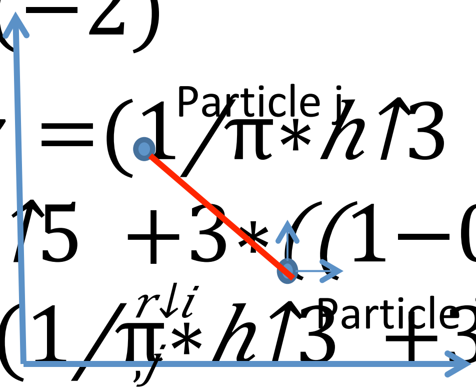
q= $\sqrt{\square} 2/2$

$$W_{j,i} = 1/\pi * h^3 * (1 + 3/2 * q^2 + 3/4 * q^3) = 1/\pi * h^3 + 3r^2/2\pi * h^5 + 3r^3/4\pi * h^6$$

Calculation of $\nabla W_{j,i}$ for Particle j

$$\partial W_{j,i}/\partial x = (1/\pi * h^3 + 3*(1^2 + (1+0.001)^2)/2\pi * h^5 + 3*(1^2 + (1+0.001)^2)^{3/2}/4\pi * h^6) - (1/\pi * h^3 + 3*2/2\pi * h^5 + 3*2^{3/2}/4\pi * h^6)/0.001 = 4.57*10^{(-2)}$$

$$\partial W_{j,i}/\partial y = (1/\pi * h^3 + 3*((1-0.001)^2 + 1^2)/2\pi * h^5 + 3*((1-0.001)^2 + 1^2)^{3/2}/4\pi * h^6) - (1/\pi * h^3 + 3*2/2\pi * h^5 + 3*2^{3/2}/4\pi * h^6)/0.001 = -4.57*10^{(-2)}$$



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Calculation of Acceleration, Velocity and Location

$$a_j = dV_j/dt = -m_i * \left(P_i/\rho_i^2 + P_j/\rho_j^2 + \prod_{j,i}\right) * \nabla W_{j,i}$$

$$= -1 * (1/1^2 + 1/1^2 + 0) * (4.57*10^{-2}, -4.57*10^{-2})$$

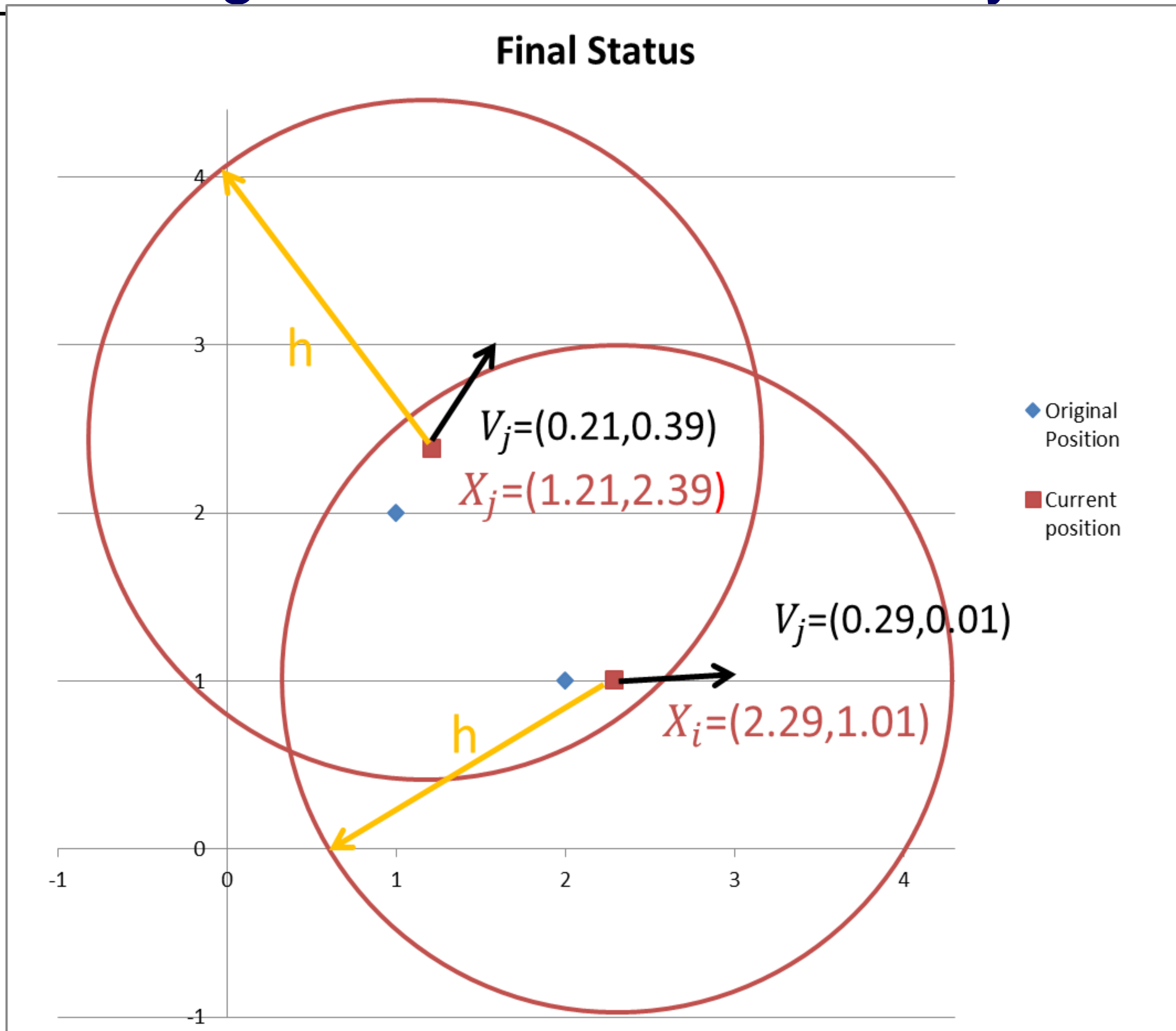
$$= (-9.14*10^{-2}, 9.14*10^{-2});$$

$$\mathbf{V_{j\,new} = V_j + a * \Delta t} = (\mathbf{0.21}, \mathbf{0.39});$$

$$\mathbf{X_{j\,new} = X_j + \Delta V_{j\,new} * \Delta t}$$

=(1.21,2.39);

The Change of Position and Velocity after Δt



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Comparing Hand Calculation with Results Obtained by the Developed Code:

Numerical Results

- t =
-     1
-
- velocity for partile: 1
-
-     0.2914   0.0086
-
- new cordinates for partile: 1
- xy =
-
-     2.2914   1.0086

- t =
-     1
-
- velocity for partile: 2
-
-     0.2086   0.3914
-
- new cordinates for partile: 2
- xy =
-
-     1.2086   2.3914

Numerical Example and Example Applications

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6. Numerical Example

- **6.1 Introduction to SPHysics**
- **6.2 Prepare for SPHysics**
- **6.3 Problem statement**
- **6.4 Input data**
- **6.5 Run model**
- **6.6 Visualization of result**

6.1 Introduction to SPHysics

Code Features:

- Open-source
- 2-D and 3-D versions
- Variable timestep
- Choices of input modes
- Visualization routines using Matlab or ParaView

6.2 Prepare for SPHysics

- Windows: Intel Visual Fortran
  Silverfrost FTN95
  GNU gfortran compiler on Cygwin

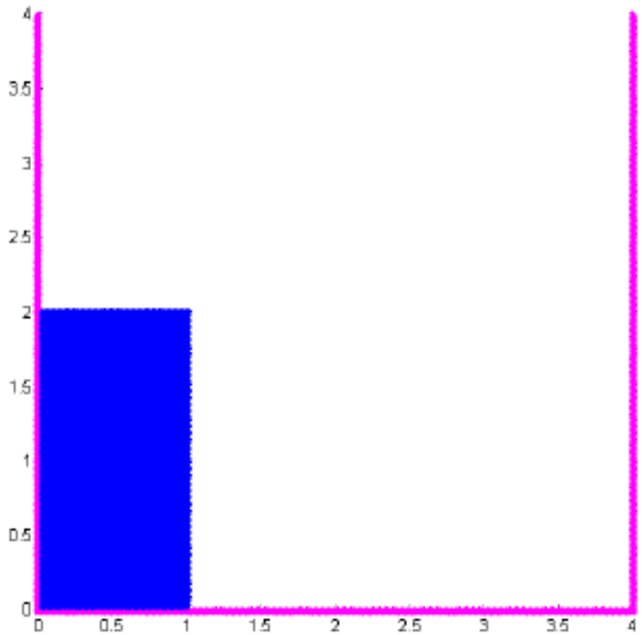
- Linux: GNU gfortran compiler
  Intel fortran compiler

- Mac: GNU gfortran

6.3 Problem statement

Water body collapse

Geometry



t=0

Properties

density = 1000 m^3/s

dx = dy = 0.03 m

dt = 0.0001 s

tmax = 3 s

6.4 Input data

6.4.1 Choose kernel function

$$A(\vec{r}) = \int A(\vec{r}\,')\,W(\vec{r} - \vec{r}\,', h)\,d\vec{r}\,'$$

- Guassian
- Quadratic
- Cubic spline
- Quintic

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6.4 Input data

6.4.2 Choose time integration scheme

- Predictor-Corrector scheme
- Verlet scheme
- Symplectic scheme
- Beeman scheme

6.4 Input data

6.4.2 Choose time integration schemes

Predictor-Corrector scheme (Average Difference operator): This scheme predicts the evolution in time as,

$$v_a^{n+1/2} = v_a^n + \frac{\Delta t}{2}\vec{F}_a^n;\ \ \rho_a^{n+1/2} = \rho_a^n + \frac{\Delta t}{2}D_a^n$$

$$r_a^{n+1/2} = r_a^n + \frac{\Delta t}{2}\vec{V}_a^n;\ \ e_a^{n+1/2} = e_a^n + \frac{\Delta t}{2}E_a^n$$

These values are then corrected using forces at the half step,

$$v_a^{n+1/2} = v_a^n + \frac{\Delta t}{2}\vec{F}_a^{n+1/2};\ \ \rho_a^{n+1/2} = \rho_a^n + \frac{\Delta t}{2}D_a^{n+1/2}$$

$$r_a^{n+1/2} = r_a^n + \frac{\Delta t}{2}\vec{V}_a^{n+1/2};\ \ e_a^{n+1/2} = e_a^n + \frac{\Delta t}{2}E_a^{n+1/2}$$

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6.4 Input data

6.4.2 Choose time integration schemes

Finally, the values are calculated at the end of the time step shown as following:

$$\bar{v}_a^{n+1} = 2\bar{v}_a^{n+1/2} - \bar{v}_a^{n};\ \rho_a^{n+1} = 2\rho_a^{n+1/2} - \rho_a^{n}$$

$$\bar{r}_a^{n+1} = 2\bar{r}_a^{n+1/2} - \bar{r}_a^{n};\ e_a^{n+1} = 2e_a^{n+1/2} - e_a^{n}$$

6.4 Input data

6.4.3 Choose options for Momentum equation

- Artificial viscosity
- Laminar
- Laminar viscosity+ Sub-Particle Scale

The artificial viscosity proposed by Monaghan (1992) has been used very often due to its simplicity.

$$\frac{d\vec{v}_a}{dt} = -\sum_b m_b \left(\frac{P_b}{\rho_b^2} + \frac{P_a}{\rho_a^2} + \Pi_{ab} \right) \vec{\nabla}_a W_{ab} + \vec{g}$$

6.4 Input data

6.4.4 Choose Density Filter:

- Zeroth Order – Shepard Filter
- First Order – Moving Least Squares (MLS)

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6.4 Input data

6.4.5 Other options

- Kernel correction
- Kernel gradient correction
- Continuity equation
- Equation of state
- Particles moving equation
- Thermal energy equation

6.5 Run model

In Linux, two main steps to run our model:

- Compile and generate SPHysicsgen_2D using SPHysicsgen.make
- Run SPHysicsgen_2D with Case1.txt as the input file
- Compile and generate SPHysics_2D using SPHysics.make
- Execute SPHysics_2D

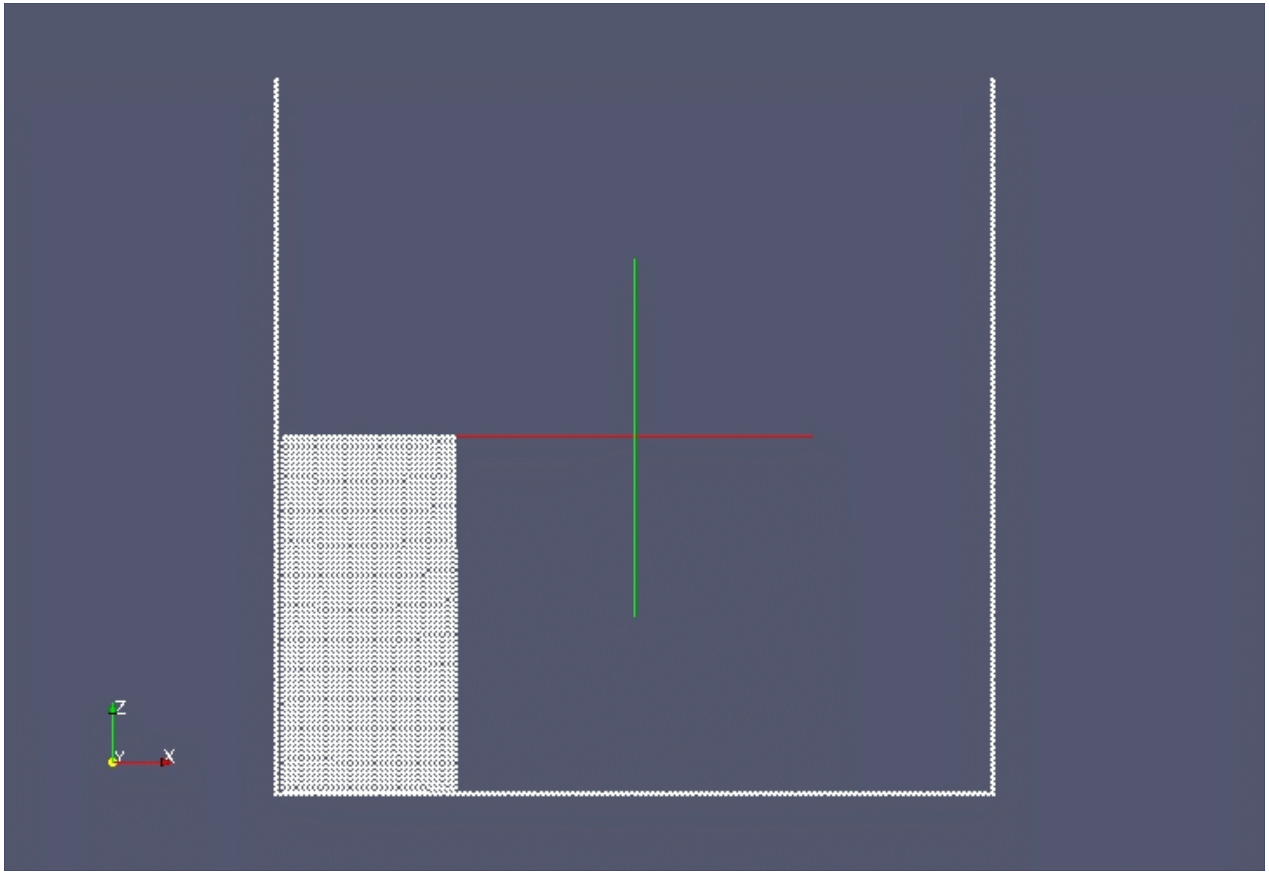
```
End
time_begin  6.9980002E-03 seconds
time_end    382.1899      seconds
Time of operation was   382.1829      seconds
```

6.4 Visualization of result

- Motion of particles

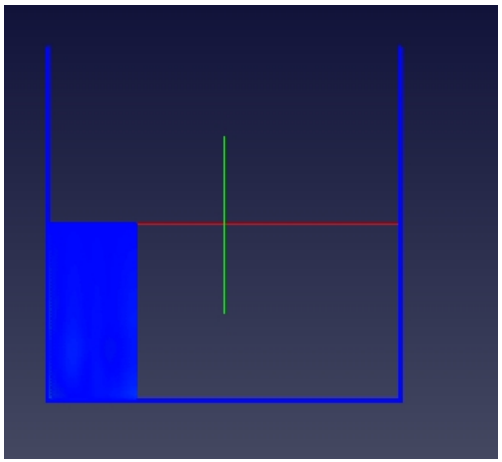
6.4 Visualization of result

- Motion of particles

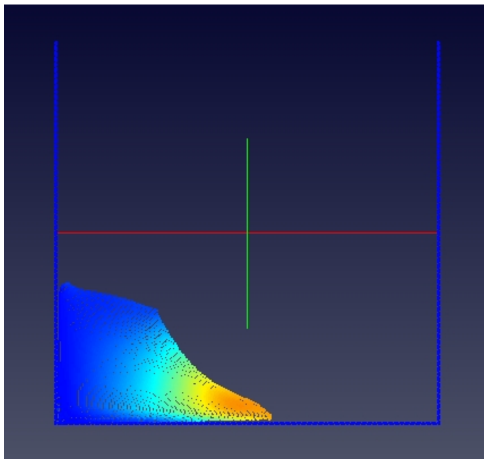


6.4 Visualization of result

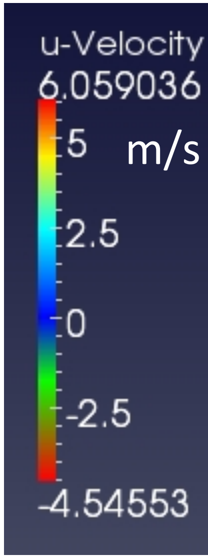
- Horizontal velocity u



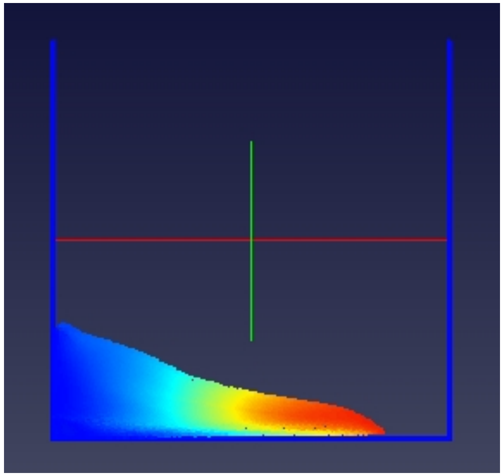
Time step=0



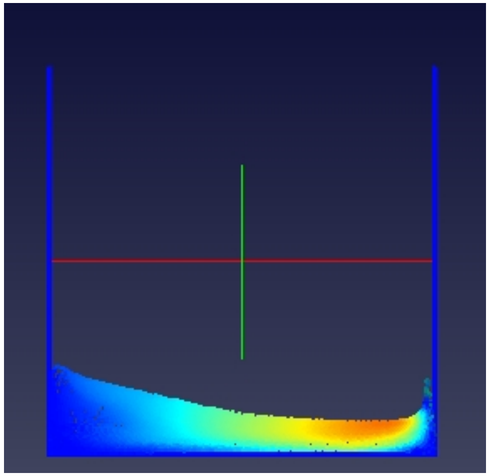
Time step=10



m/s



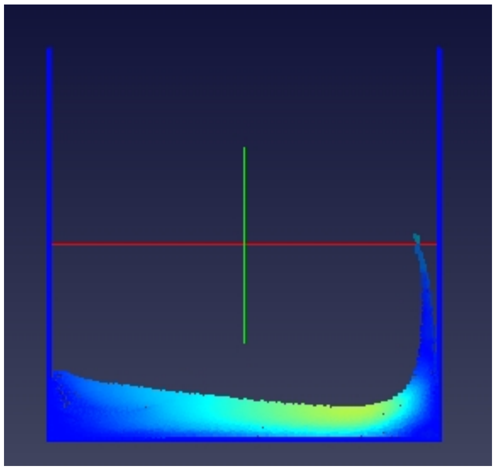
Time step=20



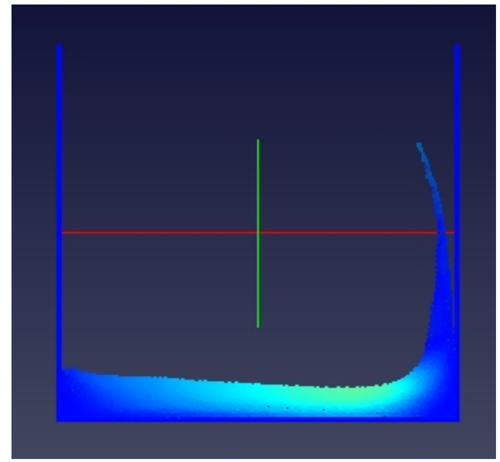
Time step=30

6.4 Visualization of result

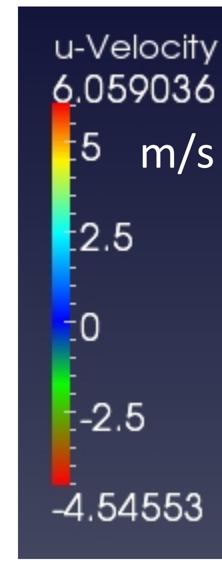
➢ Horizontal velocity u

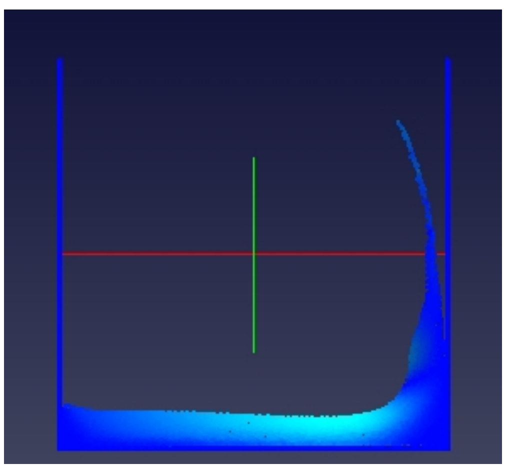


Time step=40

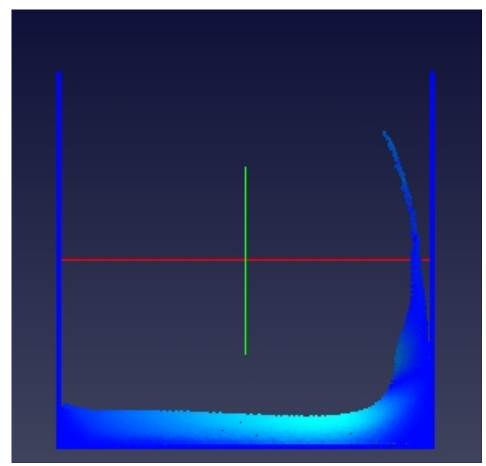


Time step=50





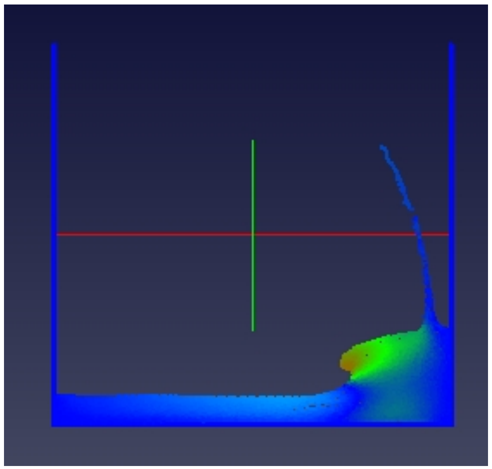
Time step=60



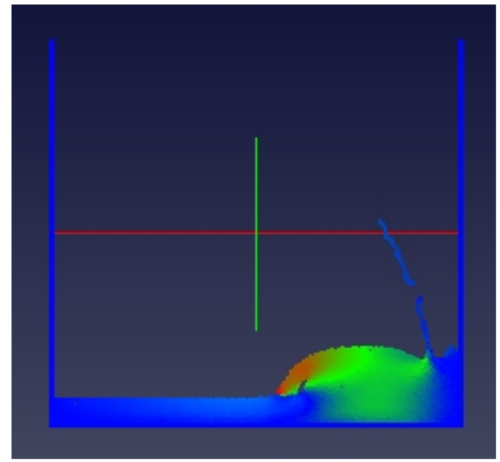
Time step=70

6.4 Visualization of result

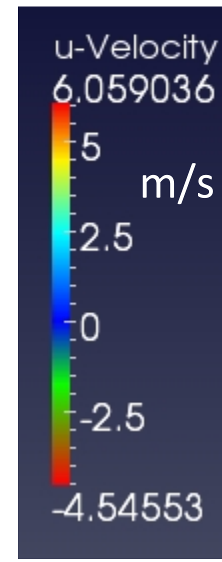
- Horizontal velocity u



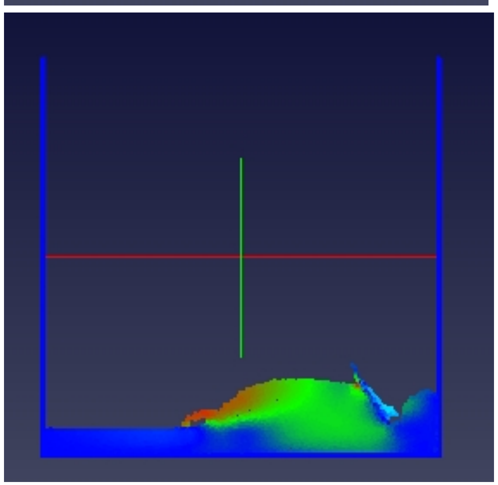
Time step=80



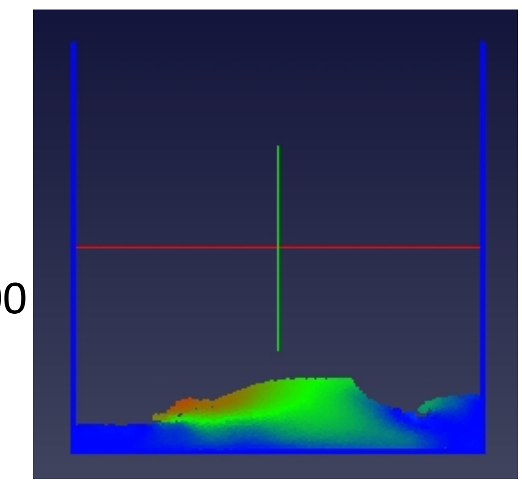
Time step=90



m/s



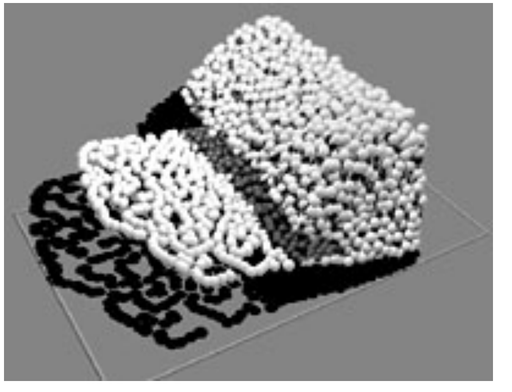
Time step=100



Time step=110

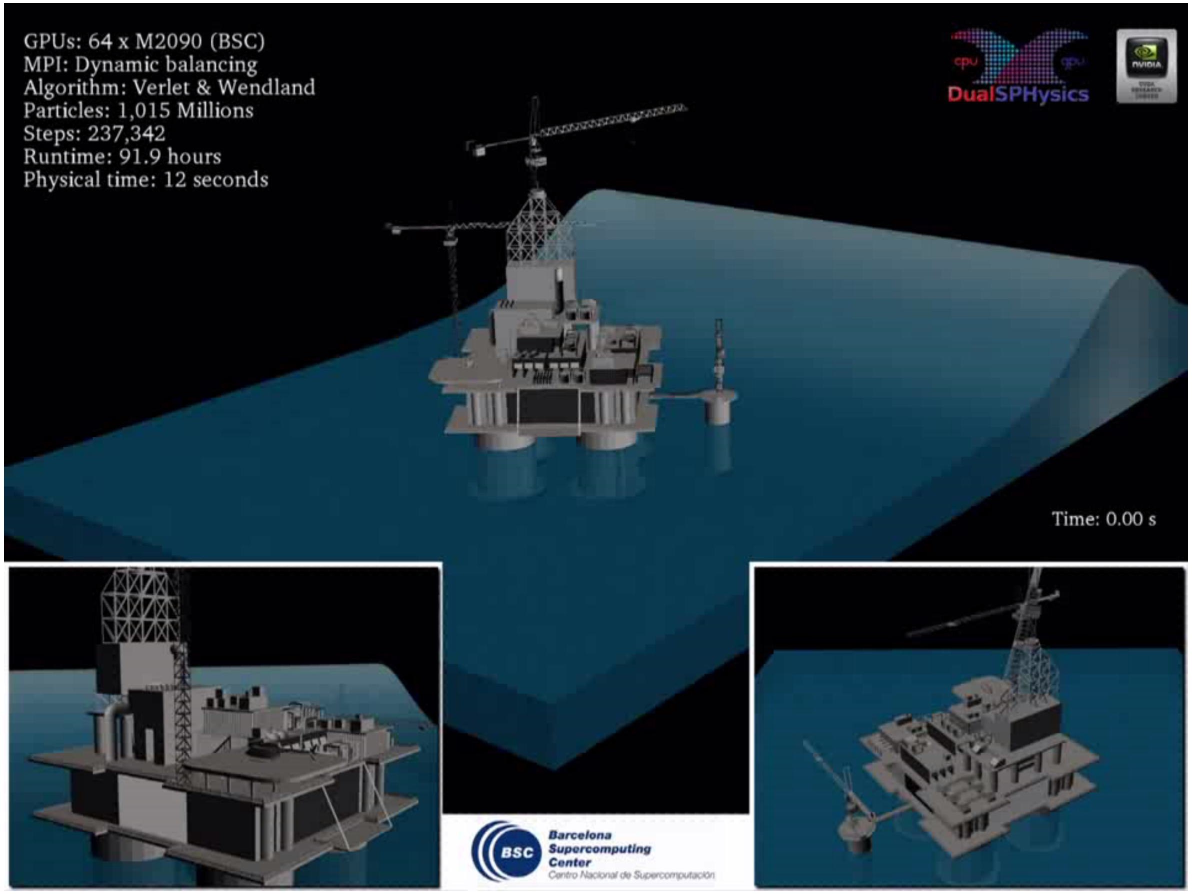
7. Numerical application

- Uses in astrophysics
- Uses in fluid simulation
- Uses in Solid mechanics



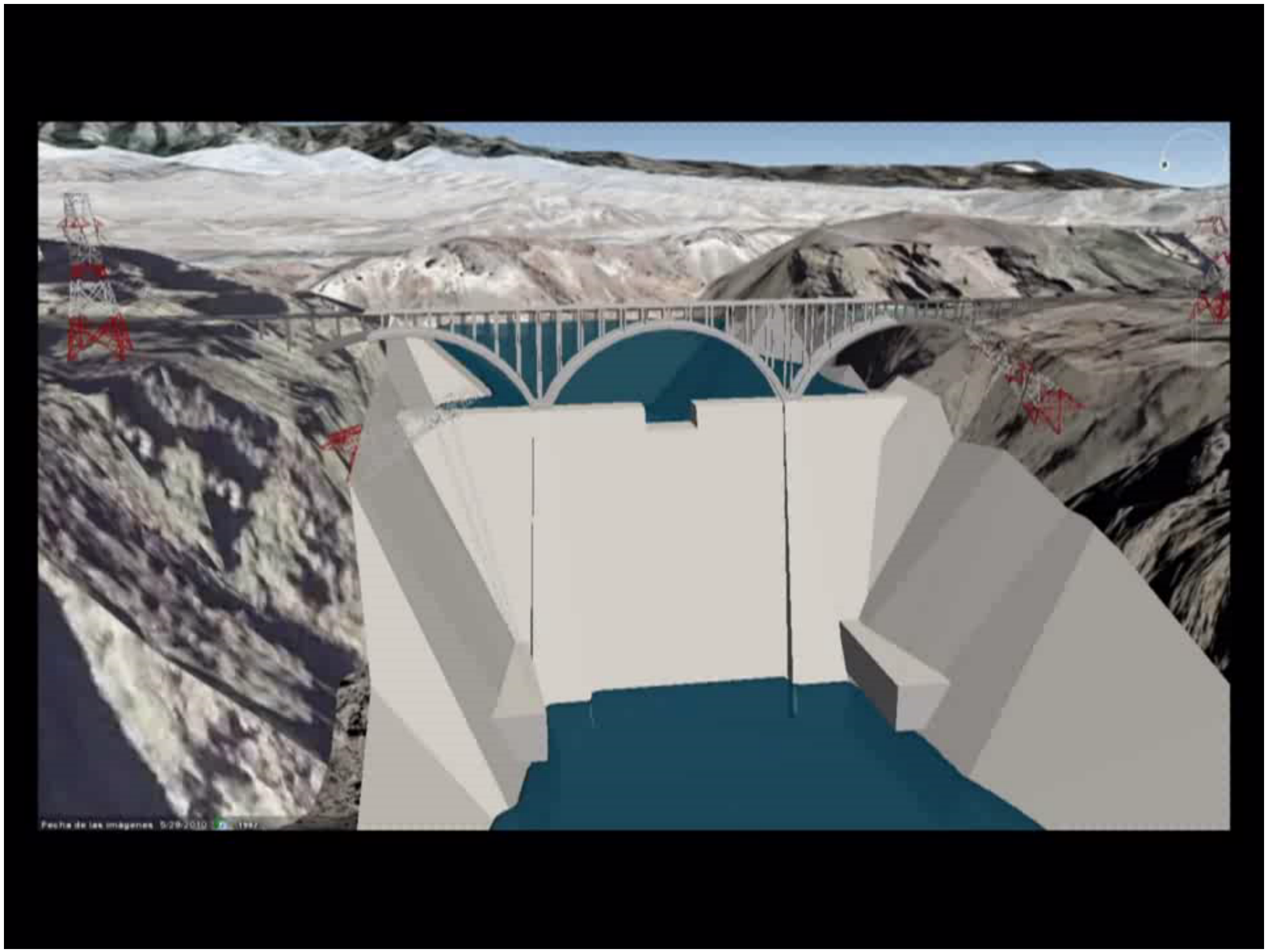
7.1 Uses in fluid simulation

Simulation of dynamic ocean wave



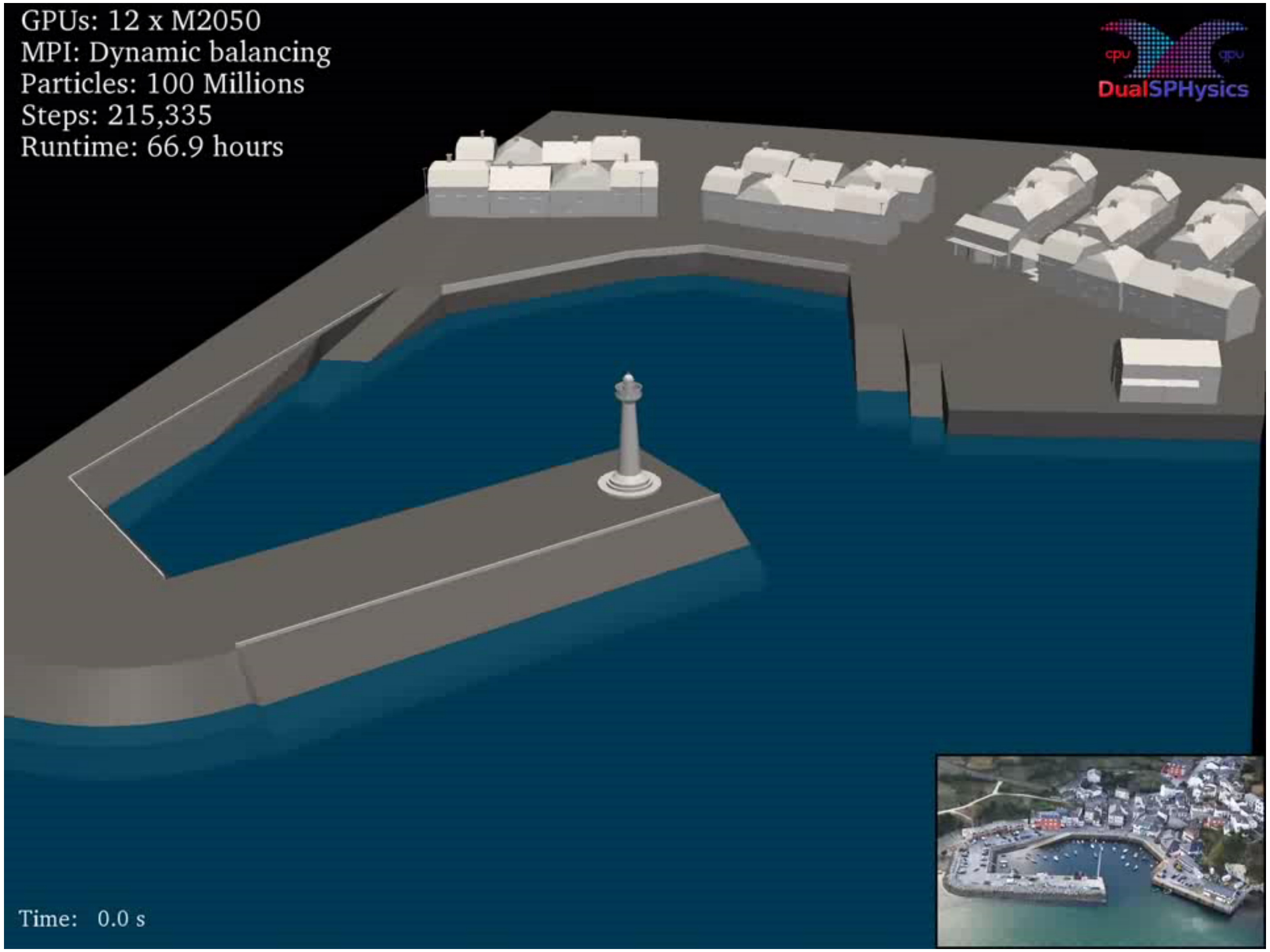
7.1 Uses in fluid simulation

Simulation for hydraulic facilities design (1)



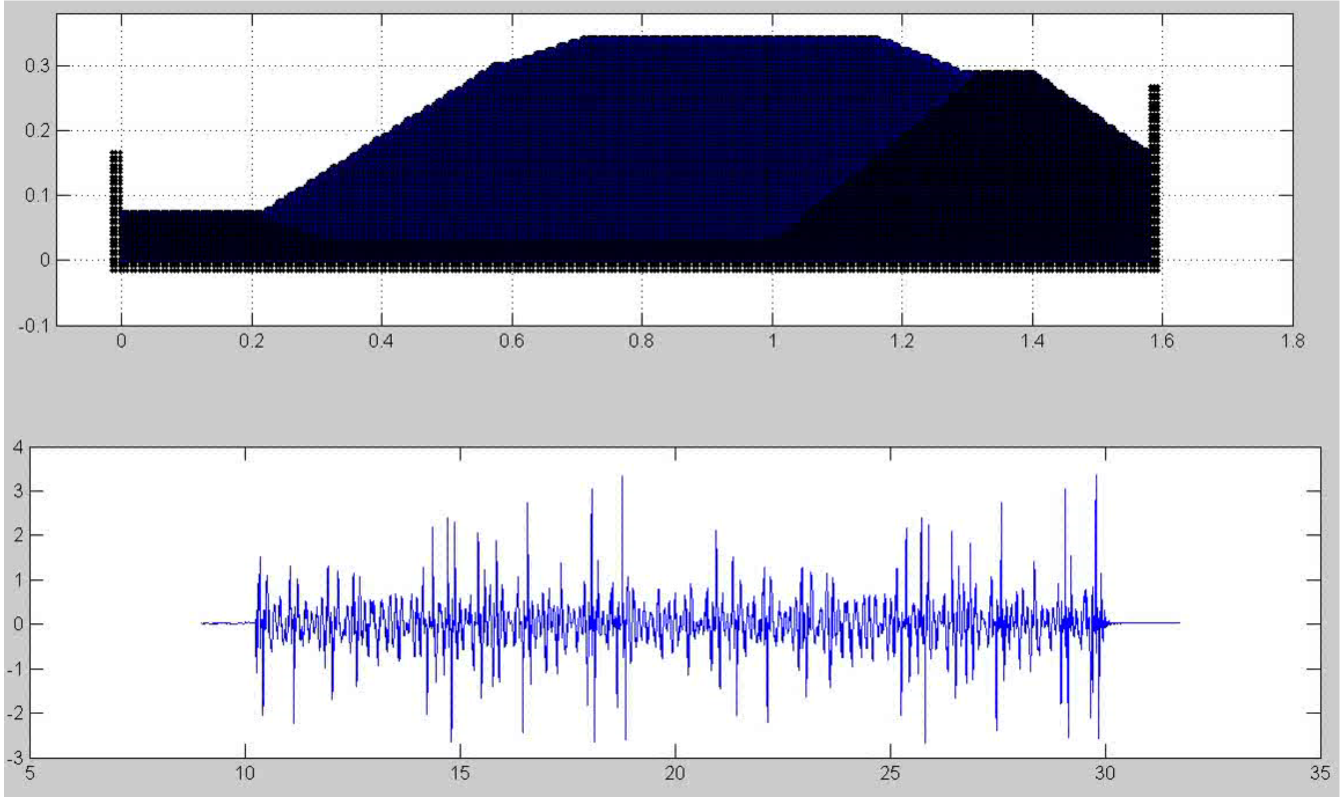
7.1 Uses in fluid simulation

Simulation for hydraulic facilities design (2)



7.2 Uses in Solid mechanics

Simulation of dam break



Thank you very much!