Boundary Element Method (BEM)

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1. Introduction

Nearly all physical phenomena occurring in nature can be describe by DEs and BCs.

Analytical Solution

- Can obtained for few specific problems with simple BCs
- Satisfy both DE and BCs

Numerical Solution

- Applicable for realistic scenario of practical engineering problems (approximate solutions)
- Satisfy one of the two (DE or BCs) and minimize the error in satisfying the other one.
- **Boundary Element Method (BEM)**: satisfy the DE exactly and minimize error in the satisfaction of BCs.

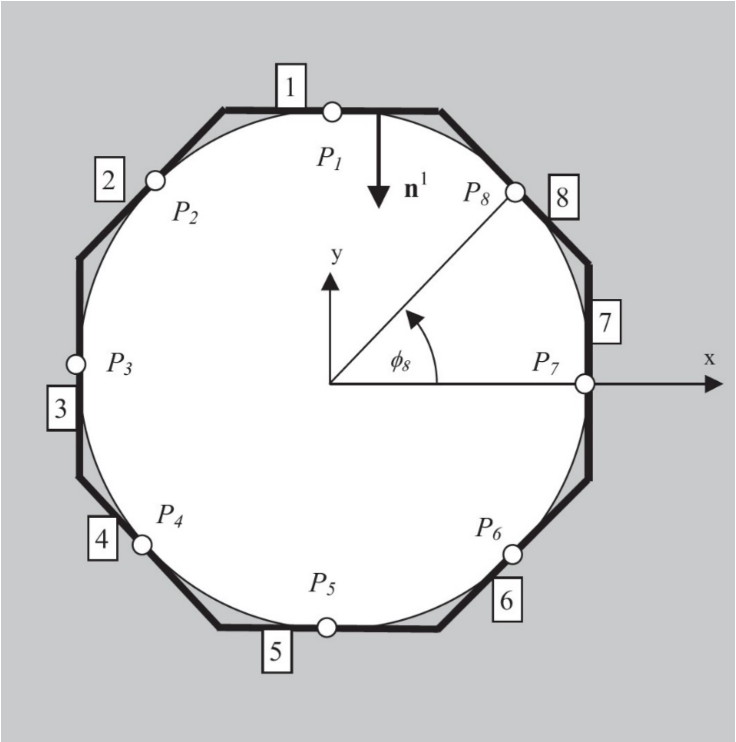
1. Introduction: The Idea of BEM

- Foundation idea of BEM came from Trefftz (1926), that we can approximate the solution to a PDE by looking at the solution to the PDE on the boundary and then use that information to find the solution inside the domain.

- As a consequence, the number of discretized elements is way less than FEM or FDM.

- BEM can be applied for potential problems governed by a DE that satisfied the Laplace equation or behaviors that has relating fundamental solutions: fluid flow, torsion of bars, diffusion and steady state heat conduction…

- Also useful for problems with complicated geometries, infinite domain problems.



Discretization into linear elements for problems of flow past cylinder [Gernot Beer et al. (2008)]

1. Introduction: Advantages of BEM

Reduction of problem dimension by 1.

- Less data preparation time.
- Easier to change the applied mesh.

Uses less number of nodes and elements.

- Faster compute time and less storage.

No approximations imposed on the solution at interior point.

- High accuracy.
- Able to modeling problems of rapidly changing stresses.

Internal points of the domain are optional.

- Filter out unwanted information, focus on section of the domain of interested.
- Further reduces compute time.

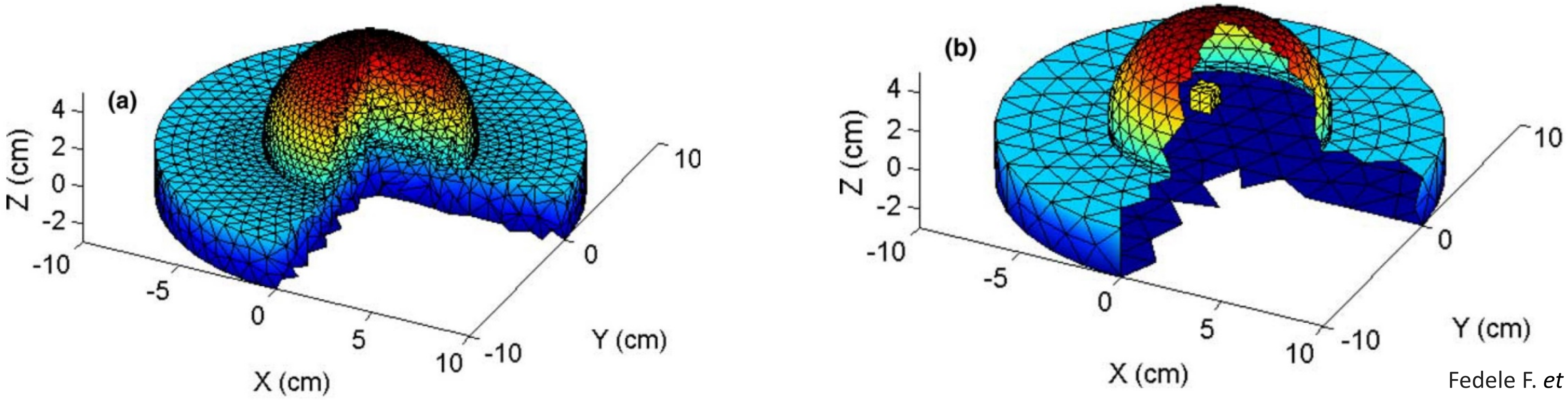
1. Introduction: Disadvantages of BEM

- For non-linear problems, the interior must be modelled, especially in non-linear material problems.
- Poor for thin structures 3-D analysis, due to large surface/volume ratio and the close proximity of nodal points on either side of the structure thickness. Causing inaccuracies in the numerical integrations.
- Requires explicit knowledge of a fundamental solution of the PDE.
- The solution matrix resulting from the BE formulation is unsymmetric and fully populated with non-zero coefficients, this means that the entire BE solution matrix must be saved in the computer core memory.

FEM vs. BEM

- Discretization of whole domain
- Good on finite domains
- Approximates interior point solution (u) & BCs solution (q) must be found from u and approximation of q may not be as accurate
- Requires no prior knowledge of solution
- Solves most linear second-order PDEs

- Discretization of boundary
- Good on infinite or semi-infinite domains
- Approximates BCs solution (q) & interior point solution (u) approximation of q is accurate
- Requires knowledge of PDE solution
- Can be difficult to solve inhomogeneous or nonlinear problems



Fedele F. *et al. (2005)*

Contents

2. Historical Perspective

- C. F. Gauss (1813)
  - Developed the Divergence Theorem.
- G. Green (1828)
  - Wrote a famous essay on the application of mathematical analysis to the theories of electricity and magnetism.

2. Historical Perspective

- E. I. Fredholm (1903)
  - Proved the existence and uniqueness of solution of the linear integral equation.

- M. A. Jaswon and A. R. Ponter (1963)
  - First formulated 2D potential problem in terms of a direct Boundary Integral Equation (BIE) and solved it numerically.

2. Historical Perspective

- F. J. Rizzo (1967)
  - Extended the work into the 2D elastostatic case.
- T. A. Cruse and F. J. Rizzo (1968)
  - Extended the work into 2D elastodynamics case.

2. Historical Perspective

- P. K. Banerjee and R. Butterfield (1975)
  - Coined the term “Boundary Element Method” in an attempt to make an analogy with Finite Element Method (FEM).
- C. A. Brebbia (1978)
  - Published the first textbook on BEM, ‘The boundary Element Method for Engineers’.
- From late 1970s, the number of journal articles shows an exponential grow rate.

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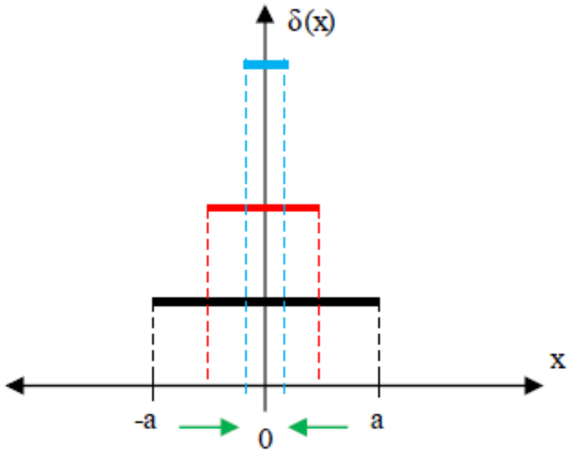
3. General Principles: Mathematical Foundations

1. Unitary impulse function

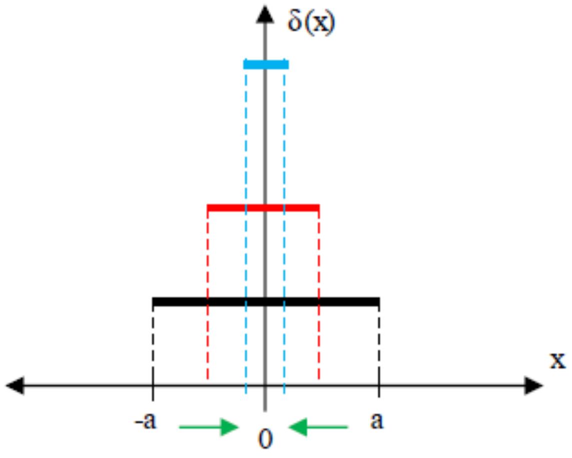


$$\delta(x) = \begin{cases} \frac{1}{2a} & , if \; -a < x < a \\ 0 & , if \; x < -a \; or \; x > a \end{cases}$$

3. General Principles: Mathematical Foundations

As we make “a” tends to zero from both sides, the function goes to infinity (called **instantaneous** impulse)

1. Unitary impulse function



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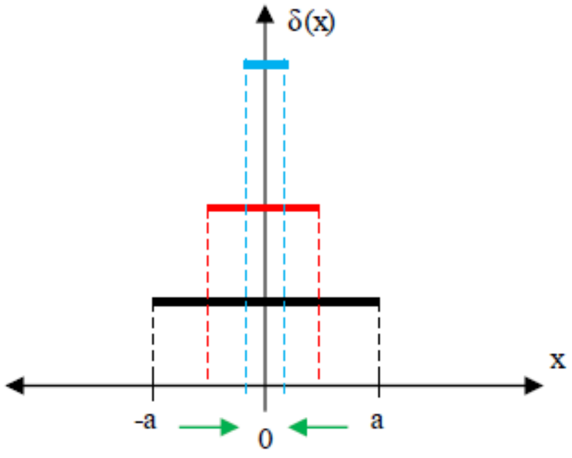

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From our definition, it possible to note that:

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1$$

The integral of such function is zero anywhere but at the location of the origin, where it is equivalent to 1

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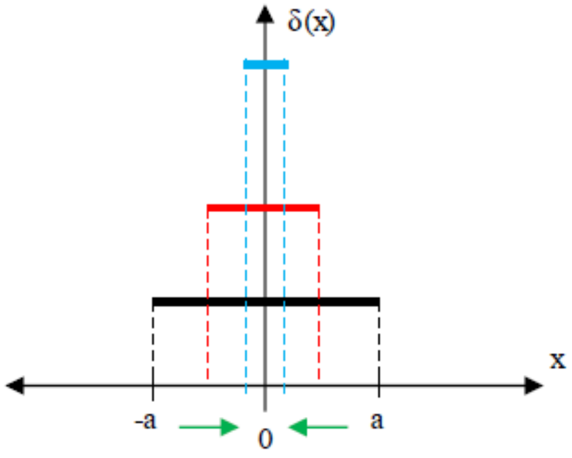

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What if impulse is not located at the origin?

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only when $x = \xi$!

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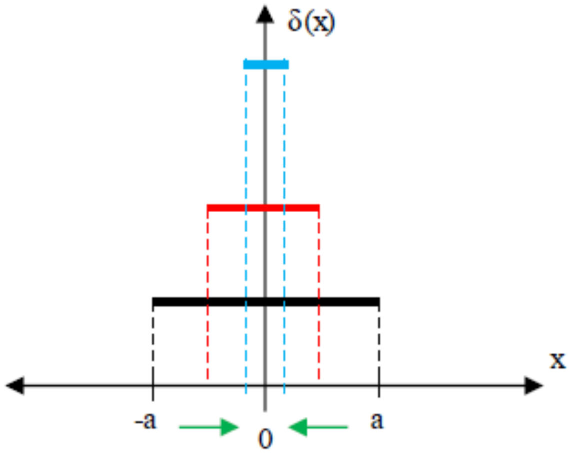

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Now we are able to understand the **sifting** property:

$$\int_{-\infty}^{+\infty} \delta(\xi - x) F(\xi) d\xi = F(x)$$

Attention: we are changing the domain of integration from $x\ to\ \xi$

3. General Principles: Mathematical Foundations

2. Green's function

Let's us explore the diffusion-type equation (linear and inhomogeneous):

$$\frac{\partial \phi}{\partial t} - D\nabla^2\phi = F \longrightarrow \textbf{Differential operator “L”} \longrightarrow L \equiv \frac{\partial(\ )}{\partial t} - \mathrm{D}\nabla^2(\ )$$

$$L(\phi) = F$$

D – diffusivity constant

ϕ – any diffusing physical quantity function of space and time

F – forcing term (source or sink) function of space and time

OBS: The IC and BC's imposed will depend on the physical system being described

3. General Principles: Mathematical Foundations

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Sometimes it is easier to find the solution to our problem by invoking an auxiliary problem:

$$L(G) = \delta(\vec{x} - \xi)\delta(t - \tau)$$

The same differential operator but now we want are seeking for the system response when subject to an instantaneous forcing term!

Green's function

3. General Principles: Mathematical Foundations

2. Green's function

Typically, we call the "**Free Space Green's Function**" the solution that only satisfies the differential operator

When working with the diffusion-type equation, the Green's function is just a Gaussian function:

$$G(x,t;\xi,\tau) = \frac{1}{D\sqrt{4\pi(t-\tau)}} exp\left[-\frac{(x-\xi)^2}{4D^2(t-\tau)}\right]$$

Caution: function is not well defined at (ξ,τ)

3. General Principles: Mathematical Foundations

2. Green's second Identity

Given the two solutions G and ϕ:

$$\iiint_V (G\nabla^2\phi - \phi\nabla^2 G)dV = \oiint_{\partial V} (G\frac{\partial\phi}{\partial n} - \phi\frac{\partial G}{\partial n})ds$$

Reducing one dimension

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4. Governing Equations

General Mass Balance Equation

$$-\nabla \cdot (\rho \vec{v}) = \frac{\partial(\rho\phi)}{\partial t}$$

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$$-\nabla \cdot (\rho\vec{v}) = \frac{\partial(\rho\phi)}{\partial t}$$

Our conventional fluid mechanics classes

$$\phi = 1$$

$$-\nabla \cdot (\rho\vec{v}) = \frac{\partial\rho}{\partial t}$$

$$\rho\nabla \cdot \vec{v} = 0$$

$$\nabla \cdot \vec{v} = 0$$

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$$\rho \nabla \cdot \vec{v} = 0$$

$$\nabla \cdot \vec{v} = 0$$

Fluid flow through porous media

We need a “bridge” that will help us to make our equation suitable to porous media flow (constitutive equation)

$$\vec{v} = -\frac{k}{\mu} \nabla P$$

4. Governing Equations

Fluid flow through porous media

$$\vec{v} = -\frac{\bar{\bar{k}}}{\mu}\nabla P \qquad -\nabla \cdot (\rho \vec{v}) = \frac{\partial(\rho\phi)}{\partial t} \longrightarrow \nabla \cdot \left(\frac{\bar{\bar{k}}}{\mu}\rho \nabla P\right) = \frac{\partial(\rho\phi)}{\partial t}$$

Darcy's Law

4. Governing Equations

Fluid flow through porous media

$$\vec{v} = -\frac{\bar{\bar{k}}}{\mu}\nabla P$$

Darcy's Law

$$-\nabla \cdot (\rho \vec{v}) = \frac{\partial(\rho\phi)}{\partial t}$$

$$\nabla \cdot \left(\frac{\bar{\bar{k}}}{\mu}\rho \nabla P\right) = \boxed{\frac{\partial(\rho\phi)}{\partial t}}$$

… working with the RHS …

$$\nabla \cdot \left(\frac{\bar{\bar{k}}}{\mu}\rho \nabla P\right) = \rho\phi c_t \frac{\partial P}{\partial t}$$

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Fluid flow through porous media

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$$-\nabla \cdot (\rho\vec{v}) = \frac{\partial(\rho\phi)}{\partial t} \longrightarrow \nabla \cdot \left(\frac{\bar{\bar{k}}}{\mu}\rho\nabla P\right) = \boxed{\frac{\partial(\rho\phi)}{\partial t}}$$

… working with the RHS …

$$\nabla \cdot \left(\frac{\bar{\bar{k}}}{\mu}\rho\nabla P\right) = \rho\phi c_t\frac{\partial P}{\partial t}$$

Homogeneous and isotropic media

Slightly-compressible fluid

Governing equation

$$\boxed{\nabla^2 P = \frac{\mu\phi c_t}{k}\frac{\partial P}{\partial t}} + \mathbf{Q}$$

Sink term (producing well)

4. Governing Equations

Integral solution protocol – the basis of Boundary Element Method

$$\nabla^2 P = \frac{\mu\phi c_t}{k}\frac{\partial P}{\partial t} + Q \xrightarrow[\text{variables}]{\text{Dimensionless}}$$

$$t_{DA} = \frac{kt}{\phi\mu c_t A}$$

$$x_D = \frac{x}{\sqrt{A}}$$

$$y_D = \frac{y}{\sqrt{A}}$$

$$p_D = \frac{p_i - p}{p_i - p_{wf}}$$

$$\longrightarrow \nabla^2 P_D - \frac{\partial P_D}{\partial t_{DA}} = Q_D$$

Writing in terms of the differential operator "L"

$$L \equiv \nabla^2(\;) - \frac{\partial(\;)}{\partial t_{DA}} \longrightarrow L(P_D) = Q_D$$

By recognizing the fundamental solution of L as "G":

$$L(G) \equiv \nabla^2 G - \frac{\partial G}{\partial t_{DA}} = \delta(\overrightarrow{X_D} - \xi)\delta(t_{DA} - \tau)$$

4. Governing Equations

$$\left(\nabla^2 P_D - \frac{\partial P_D}{\partial t_{DA}} = Q_D \right) \times$$ **Green's function**

George Green
(1793 – 1841)

Fundamental solution (Green's function)

Green's second identity

Johann C. F. Gauss
(1777 – 1855)

Divergence (Gauss) theorem

$$P_D(\overrightarrow{X_D}, t_{DA}) = \int_0^{t_{DA}} \int_{\partial D} \left[G \frac{\partial P_D}{\partial n} - P_D \frac{\partial G}{\partial n} \right] ds d\tau - \int_0^{t_{DA}} \int_{\partial D} G Q_D d\xi d\tau$$

Valid within the domain "D"

Usually either P_D or $\frac{\partial P_D}{\partial n}$ are specified as BC's

Boundary Element Method Protocol

4. Governing Equations

Boundary Element Method

Every point within the domain (including the boundary) could be represented by a boundary integral solution with the Green function (differential equation is linear).

In general form the boundary integral solution can be represented by:

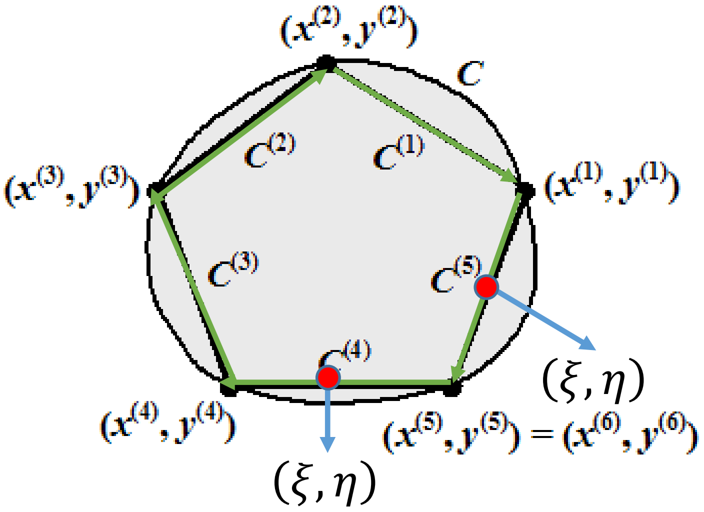
$$\beta(\xi,\eta)\phi(x,y) = \oint_{\partial V} \left[\frac{\partial G(x,y;\xi,\eta)}{\partial n}\phi(x,y) - G(x,y;\xi,\eta)\frac{\partial \phi(x,y)}{\partial n}\right] ds$$

$$\beta = \begin{cases} 1, if\,(\xi,\eta) is\ inside\ domain\ \Omega \\ \frac{1}{2}, if\,(\xi,\eta) is\ on\ smooth\ boundary\ \partial\Omega \end{cases}$$

4. Governing Equations

Boundary Element Method

$$\beta(\xi,\eta)\phi(x,y) = \int_{\partial c} \left[\frac{\partial G(x,y;\xi,\eta)}{\partial n}\phi(x,y) - G(x,y;\xi,\eta)\frac{\partial \phi(x,y)}{\partial n}\right] ds$$



- The value of random point requires 2 boundary conditions from each boundary element;
- N boundary elements, each boundary elements has 2 boundary conditions (one is prescribed, and one is unknown);
- N unknowns require N equations!

Based on prescribed conditions, fix fictitious point at one boundary for Green function, obtain the boundary integral

Move fictitious point to other boundaries, repeat the previous step to obtain boundary integrals

Based on the boundary integrals, N equations to solve N unknows (unprescribed boundary conditions)

Based on all the boundary conditions, value at each point could be obtained

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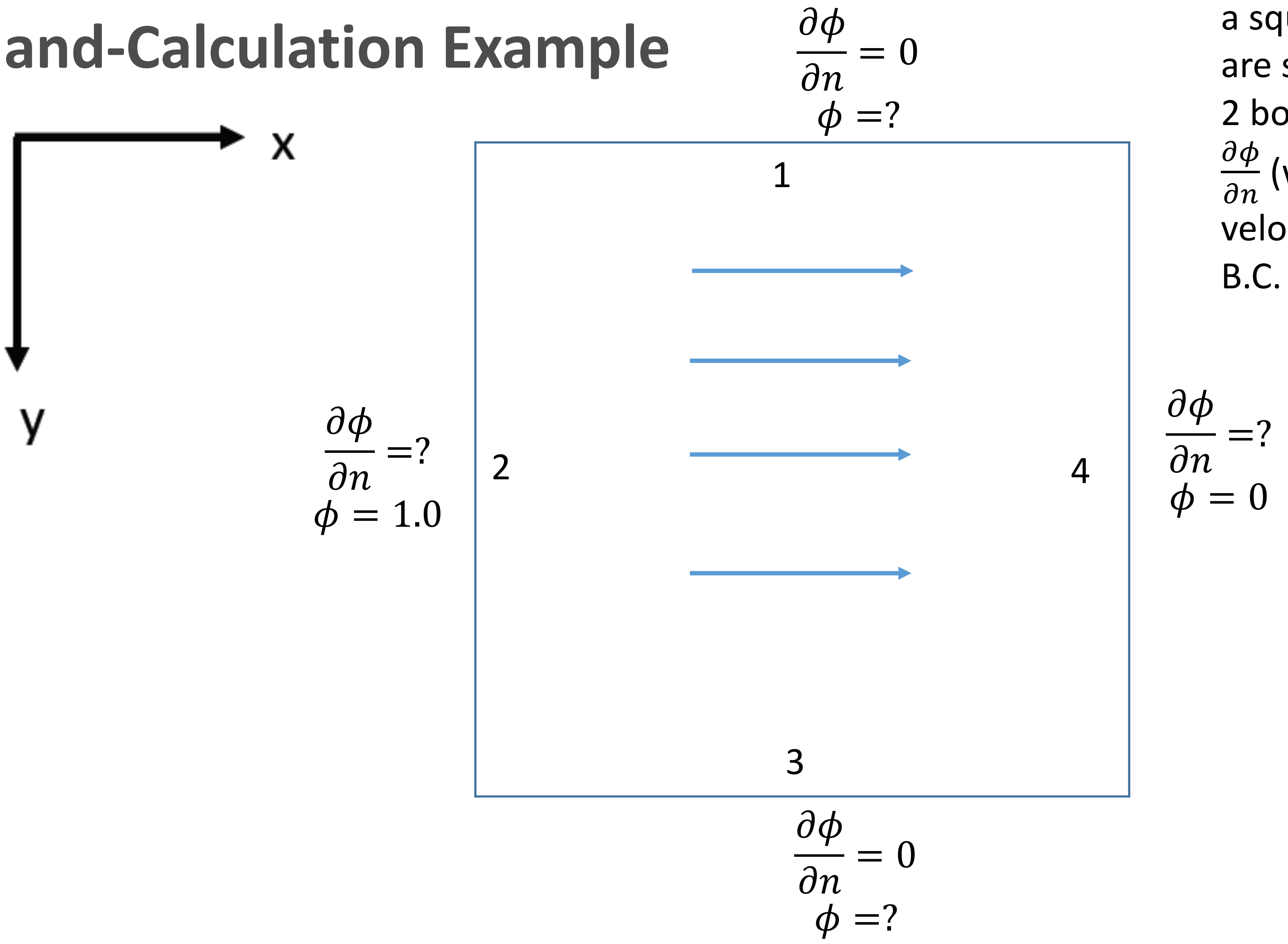
5. Hand-Calculation Example



Imagine inside a fluid flow field and a square domain exists. 2 boundaries are specified with flow potential and 2 boundaries are specified with the $\frac{\partial \phi}{\partial n}$ (which could be considered as velocity given). What is the other B.C. on each boundary?

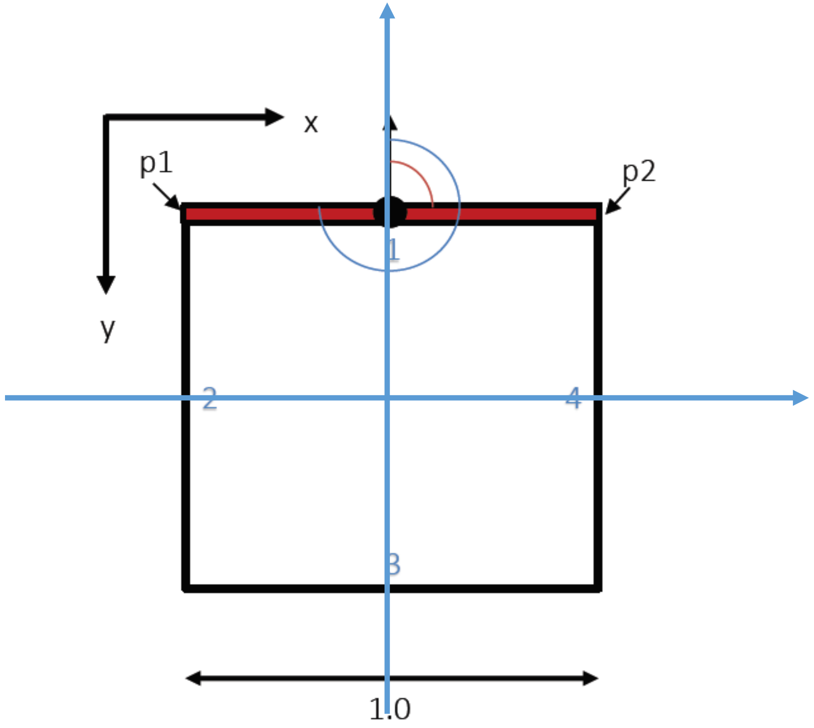
4 boundaries; 4 B.C. given

5. Hand-Calculation Example

$$\oint_{\partial V}\left[\left(\frac{\partial G(x,y;\xi,\eta)}{\partial n}-\beta(\xi,\eta)\right)\phi(x,y)\right]ds(x,y)=\oint_{\partial V}\left[G(x,y;\xi,\eta)\frac{\partial\phi(x,y)}{\partial n}\right]ds(x,y)$$

$$If\begin{cases} G(x,y;\xi,\eta)=\frac{1}{2\pi}(x(\ln(r)-x+y\theta)\Big|_1^2 \\ V_x=\frac{\partial G}{\partial x}=-\frac{k}{2\pi}(\ln(r))\Big|_1^2 \\ V_y=\frac{\partial G}{\partial y}=-\frac{k}{2\pi}(ln\theta)\Big|_1^2 \\ r=\sqrt{(x-\xi)^2+(y-\eta)^2} \\ k=const=10 \end{cases}$$

$$G_{11}(0{,}0.5;0{,}0.5)=\frac{1}{2\pi}(0.5(\ln(0.5)-0.5)-(-0.5(\ln(0.5)+0.5)]$$
$$=-0.27$$



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$$\oint_{\partial V}\left[\left(\frac{\partial G(x,y;\xi,\eta)}{\partial n}-\beta(\xi,\eta)\right)\phi(x,y)\right]ds(x,y)=\oint_{\partial V}\left[G(x,y;\xi,\eta)\frac{\partial\phi(x,y)}{\partial n}\right]ds(x,y)$$

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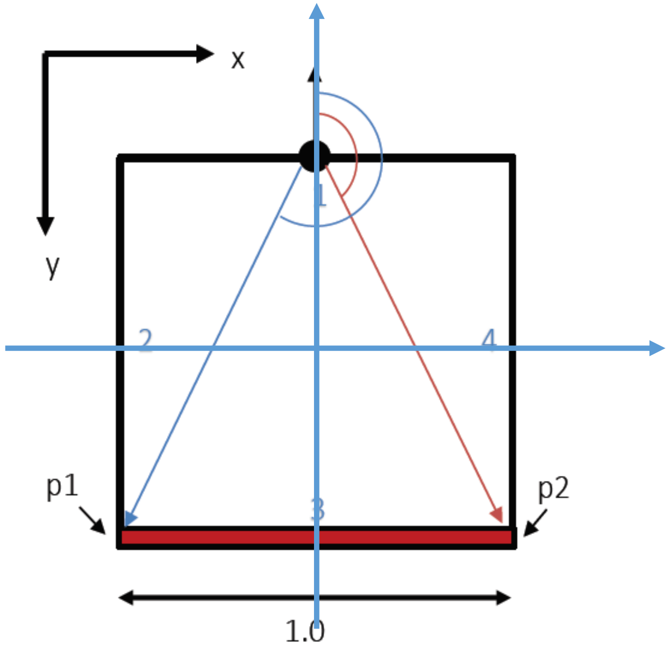

$$G_{31}(0,-0.5;0{,}0.5)$$
$$=\frac{1}{2\pi}\left[0.5\left(\ln\left(\frac{\sqrt{5}}{2}\right)-0.5-1.117-(-0.5\ln\left(\frac{\sqrt{5}}{2}\right)+0.5-2.034\right)\right]=0.0062$$

5. Hand-Calculation Example

$$\oint_{\partial V}\left[\left(\frac{\partial G(x,y;\xi,\eta)}{\partial n}-\beta(\xi,\eta)\right)\phi(x,y)\right]ds(x,y)=\oint_{\partial V}\left[G(x,y;\xi,\eta)\frac{\partial\phi(x,y)}{\partial n}\right]ds(x,y)$$

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$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix}$$

$$=\begin{bmatrix} -0.27 & -0.053 & 0.006 & -0.053 \\ -0.053 & -0.27 & -0.053 & 0.006 \\ 0.006 & -0.053 & -0.27 & -0.053 \\ -0.053 & 0.006 & -0.053 & -0.27 \end{bmatrix}$$

5. Hand-Calculation Example

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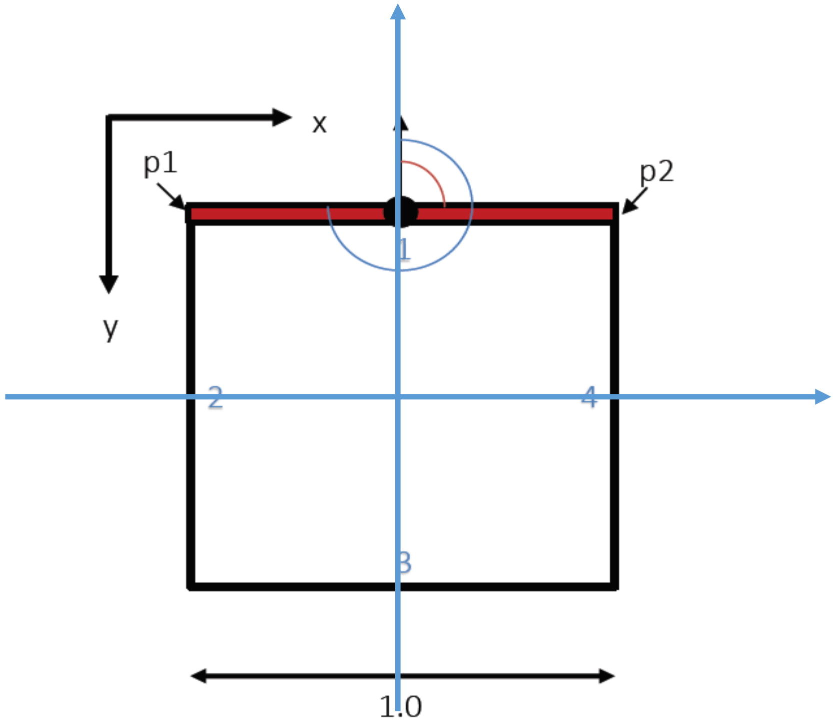
$$V_{11}(0,0.5;0,0.5)=\frac{10}{2\pi}\left(\frac{\pi}{2}-\frac{3}{2\pi}\right)=5.00$$



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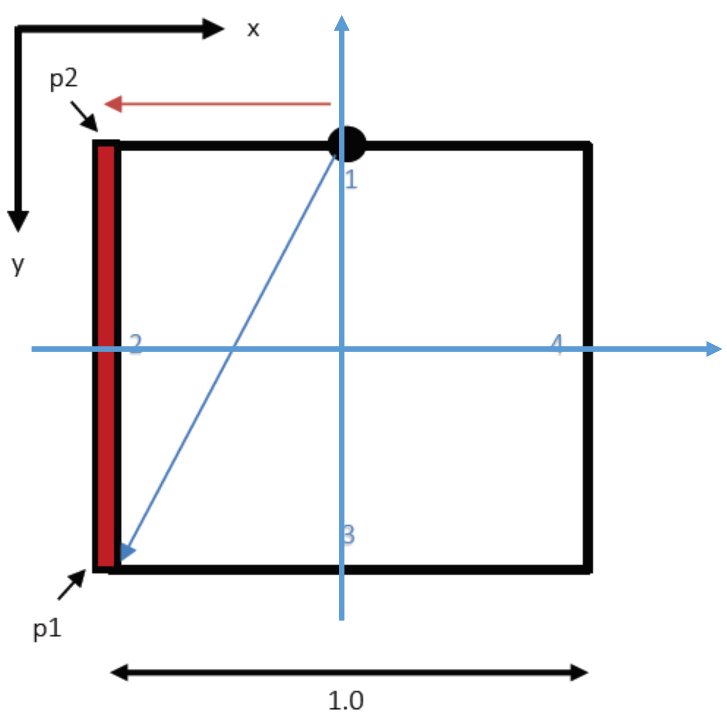
$$V_{12}(0,-0.5;0{,}0.5)=\frac{10}{2\pi}\left[\ln 0.5-\ln\left(\frac{\sqrt{5}}{2}\right)\right]=-1.281$$



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$$\oint_{\partial V}\left[\left(\frac{\partial G(x,y;\xi,\eta)}{\partial n}-\beta(\xi,\eta)\right)\phi(x,y)\right]ds(x,y)=\oint_{\partial V}\left[G(x,y;\xi,\eta)\frac{\partial \phi(x,y)}{\partial n}\right]ds(x,y)$$

$$If\begin{cases} G(x,y;\xi,\eta)=\dfrac{1}{2\pi}\left(x(\ln(r)-x+y\theta)\right)\Big|_1^2 \\ V=\dfrac{\partial G(x,y;\xi,\eta)}{\partial n}=-\dfrac{k}{2\pi}(\ln(r))\Big|_1^2 \\ r=\sqrt{(x-\xi)^2+(y-\eta)^2} \\ k=const=10 \end{cases}$$

$$\begin{bmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{bmatrix}$$

$$=\begin{bmatrix} 5.000 & -1.281 & -1.476 & -1.281 \\ -1.281 & 5.000 & -1.281 & -1.476 \\ -1.476 & -1.281 & 5.000 & -1.281 \\ -1.281 & -1.476 & -1.281 & 5.000 \end{bmatrix}$$

5. Hand-Calculation Example

$$\begin{bmatrix} V_{11}-\beta & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22}-\beta & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33}-\beta & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44}-\beta \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial \phi}{\partial n}\right)_1 \\ \left(\frac{\partial \phi}{\partial n}\right)_2 \\ \left(\frac{\partial \phi}{\partial n}\right)_3 \\ \left(\frac{\partial \phi}{\partial n}\right)_4 \end{bmatrix}$$

$$\begin{bmatrix} \phi_1 \\ \left(\frac{\partial \phi}{\partial n}\right)_2 \\ \phi_3 \\ \left(\frac{\partial \phi}{\partial n}\right)_4 \end{bmatrix} = \begin{bmatrix} 0.409 \\ -16.284 \\ 0.409 \\ 8.987 \end{bmatrix}$$

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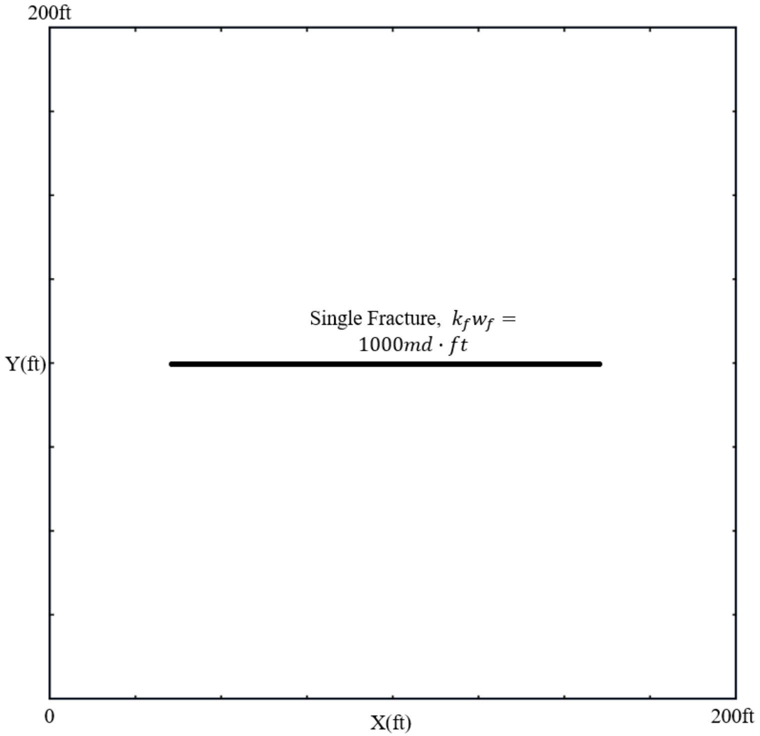
6. Numerical Example

- The general boundary integral solution

$$\lambda(\xi,\eta)\,\overline{p_D}(\xi,\eta;s) = \int_{\partial\Omega}\left(\overline{G}\,\frac{\partial\,\overline{p_D}}{\partial\boldsymbol{n}} - \overline{p_D}\,\frac{\partial\,\overline{G}}{\partial\boldsymbol{n}}\right)dS + \int_{\Omega}\overline{G}\,\overline{Q_D}\,d\omega$$



Mesh free!

Reservoir and Fluid Properties		
Reservoir dimensions x,y		200ft
Initial pressure p_i		5000psia
Thickness h		50ft
Permeability k		0.01md
Pore Compressibility c_ϕ		10^{-12} 1/psi
Porosity ϕ		0.3
Fluid viscosity μ		0.32cp
Fluid compressibility c_f		3×10^{-6}1/psi
Fracture half length x_f		50ft
Bottomhole pressure p_{wf}		1000psia
Fracture Conductivity $k_f w_f$	Case A	5 md-ft
	Case B	10 md-ft
	Case C	50 md-ft

Oil single phase;

BEM-FVM combined method!

Reference: Zhang, M., & Ayala, L. F. (2018, September). Application of Superposition Principle to Variable Rate/Pressure Production Analysis in Unconventional Gas Reservoirs. In *SPE Annual Technical Conference and Exhibition*. Society of Petroleum Engineers.

6. Numerical Example

Application: Slightly-compressible fluid flow (Oil) with fractures

$$\nabla^2 p_D = \frac{\partial p_D}{\partial t_D} - Q_D(t_D)$$

Laplace Transform

$$(\nabla^2 - s)\overline{p_D} + \overline{Q_D} = 0$$

Multiply by the Green's function and integrate over the space domain

$$\int_\Omega \left[\overline{G}(\nabla^2 - s)\overline{p_D} - \overline{G}\,\overline{Q_D}\right] d\omega = 0$$

Expanding

$$-\int_\Omega \nabla \cdot \left(\overline{G}\nabla\,\overline{p_D} - \overline{p_D}\nabla\,\overline{G}\right) d\omega + \int_\Omega \overline{p_D}\left(\nabla^2\,\overline{G} - s\,\overline{G}\right) d\omega - \int_\Omega \overline{G}\,\overline{Q_D}\, d\omega = 0$$

Single BEM would not solve this problem; the fractures need special treatment, e.g. FVM!

Application: Slightly-compressible fluid flow (Oil) with fractures

$$= \boxed{-\int_{\Omega} \nabla \cdot \left(\overline{G} \nabla \overline{p_D} - \overline{p_D} \nabla \overline{G} \right) d\omega} + \boxed{\int_{\Omega} \overline{p_D} \left(\nabla^2 \overline{G} - s\,\overline{G} \right) d\omega} - \int_{\Omega} \overline{G}\,\overline{Q_D}\, d\omega = 0$$

First term:

$$-\int_{\Omega} \nabla \cdot \left(\overline{G} \nabla \overline{p_D} - \overline{p_D} \nabla \overline{G} \right) d\omega$$

$$= -\int_{\partial\Omega} \left(\overline{G} \nabla \overline{p_D} - \overline{p_D} \nabla \overline{G} \right) \cdot \boldsymbol{n}\, dS$$

Second term:

$$\int_{\Omega} \overline{p_D} \boxed{\left(\nabla^2 \overline{G} - s\,\overline{G} \right)} d\omega$$

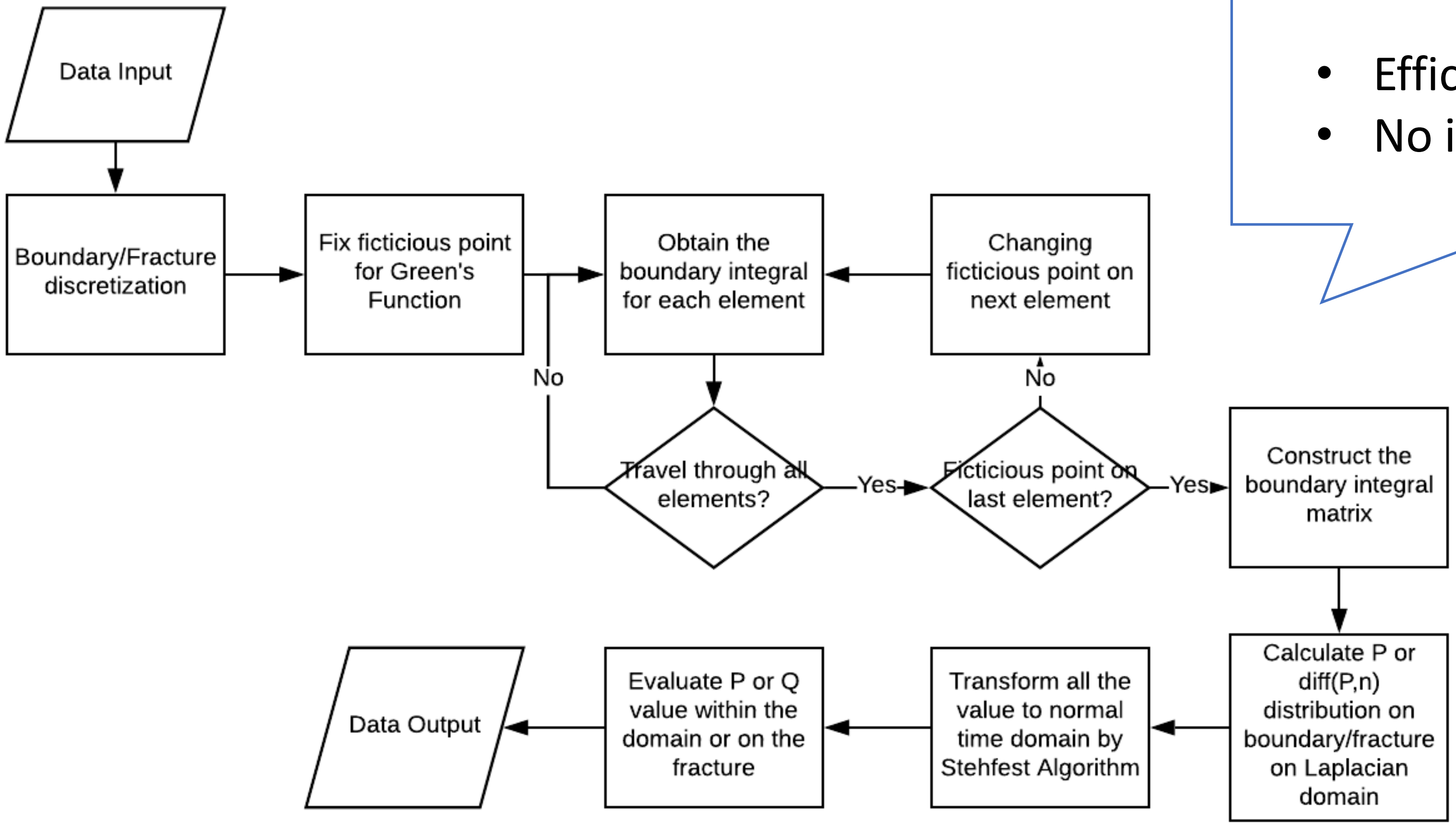
$$= -\int_{\Omega} \overline{p_D} \boxed{\delta\left(x_D - x_{D_0}\right) \delta\left(y_D - y_{D_0}\right)} d\omega = -\overline{p_D}(\xi, \eta; s)$$

Boundary integral solution in Laplace domain:

$$\beta(\xi, \eta)\, \overline{p_D}(\xi, \eta; s) = \int_{\partial\Omega} \left(\overline{G} \frac{\partial\, \overline{p_D}}{\partial \boldsymbol{n}} - \overline{p_D} \frac{\partial\, \overline{G}}{\partial \boldsymbol{n}} \right) dS + \int_{\Omega} \overline{G}\,\overline{Q_D}\, d\omega$$

$$\beta(\xi, \eta) = \begin{cases} 1, if\, (\xi, \eta) is\ inside\ domain\ \Omega \\ \frac{1}{2}, if\, (\xi, \eta) is\ on\ smooth\ boundary\ \partial\Omega \end{cases}$$

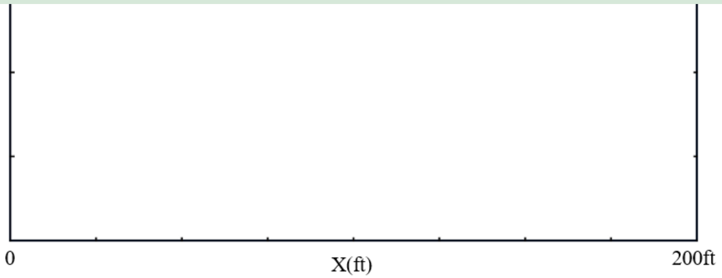
6. Numerical Example



- Efficient and implicit;
- No iteration is required!

Flow chart for oil single phase using Boundary Element Method

```
for m=1:Nb
    for i=1:Nb
        [coordinate_m]=FIND(m,loca_dimless);
        [IA,IB,~,~,~]=Int_I(loca_dimless,coordinate_m(5:6),i,0)
        BP(m,i)=IA-1/2*delta(m,i);
        SBx(i)=IB*dPd(i);
        BEMFVM(m,i)=BP(m,i);
        count=count+1
    end
    SB(m)=-sum(SBx);
end
% BQ
for m=1:Nb
    for j=Nb+1:Nb+Nf    % when storing the data, fracture locations staring f
        [coordinate_m]=FIND(m,loca_dimless);
        [~,~,IQ,~,~]=Int_I(loca_dimless,coordinate_m(5:6),j,1)
        BQ(m,j-Nb)=IQ;
        BEMFVM(m,j)=BQ(m,j-Nb);
    end
end
% I and CQ
for n=Nb+1:Nb+Nf
    for j=Nb+1:Nb+Nf
        I(n-Nb,n-Nb)=1;
        [coordinate_n]=FIND(n,loca_dimless);
        [~,~,IQ,~,~]=Int_I(loca_dimless,coordinate_n(5:6),j,1)
        CQ(n-Nb,j-Nb)=IQ;
        BEMFVM(n,j)=CQ(n-Nb,j-Nb);
        BEMFVM(n,n+Nf)=I(n-Nb,n-Nb);
    end
end
```



Reference: Z
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Analysis in U
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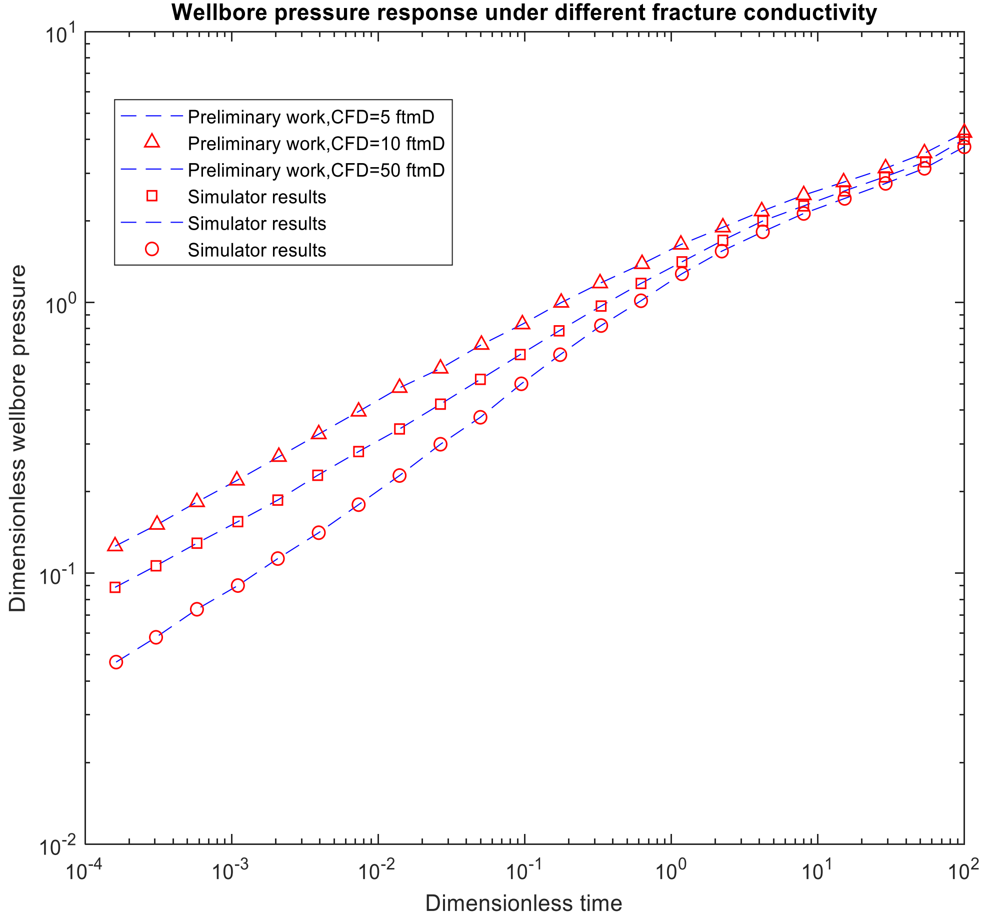
```
% CP and SC
for n=Nb+1:Nb+Nf
    for i=1:Nb
        [coordinate_n]=FIND(n,loca_dimless);
        [IA,IB,~,~,~]=Int_I(loca_dimless,coordinate_n(5:6),i,0)
        CP(n-Nb,i)=IA;
        SCx(i)=IB*dPd(i);
        BEMFVM(n,i)=CP(n-Nb,i);
    end
    SC(n)=-sum(SCx);
end
% FP and FQ

[FP,SF]=Frac_T(dlfD,index,Nf,PDwf);
for n=1:Nf
    for j=1:Nf
        FQ(n,j)=k/kfwf;
        BEMFVM(n+Nb+Nf,j+Nb)=FQ(n,j);
        BEMFVM(n+Nb+Nf,j+Nb+Nf)=FP(n,j);
    end
end

S(1:Nb,1)=SB;
S((Nb+1):(Nb+Nf),1)=SC(1:Nf,1);
S((Nb+Nf+1):(Nb+2*Nf),1)=SF(1:Nf,1);



% To build the large matrix


  function [tag]=delta(m,i)
        if m==i
            tag=1;
        else
            tag=0;
        end
    end
```

6. Numerical Example



Reservoir and Fluid Properties		
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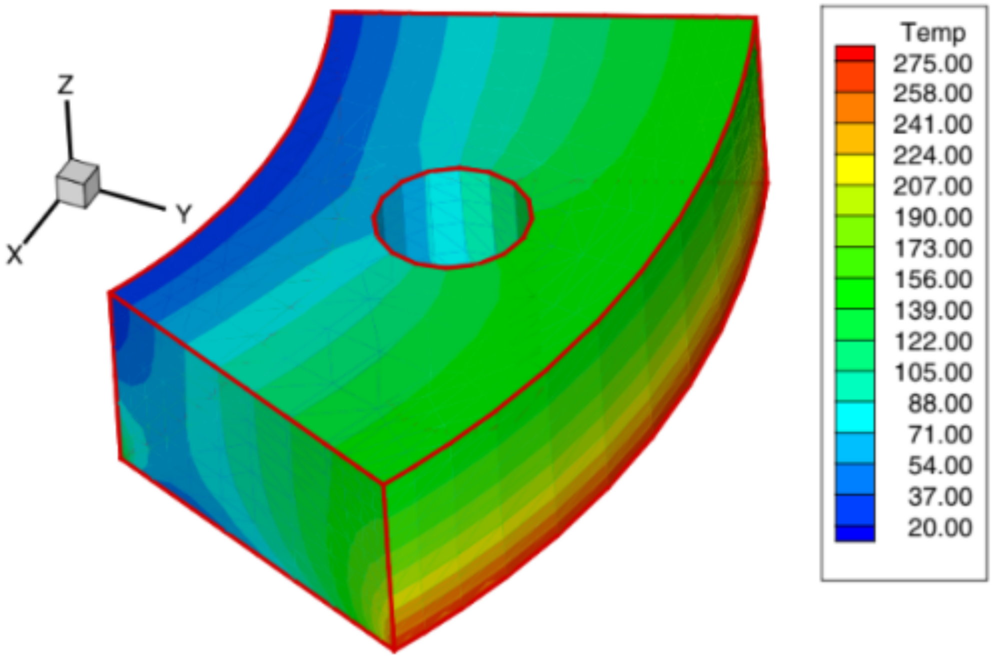
Thickness h		50ft
Permeability k		0.01md
Pore Compressibility c_ϕ		10^{-12} 1/psi
Porosity ϕ		0.3
Fluid viscosity μ		0.32cp
Fluid compressibility c_f		3×10^{-6}1/psi
Fracture half length x_f		50ft
Bottomhole pressure p_{wf}		1000psia
Fracture Conductivity $k_f w_f$	Case A	5 md-ft
	Case B	10 md-ft
	Case C	50 md-ft

- Good agreement between results with BEM and in-house FVM simulators'!

Contents

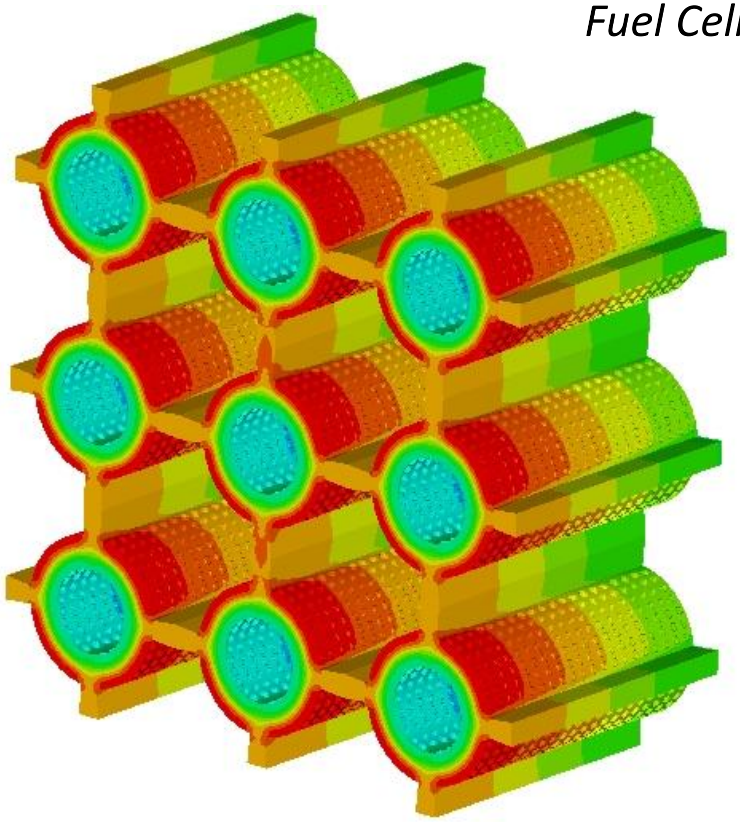
7. Example Applications

Thermal Analysis



Sutradhar et al. (2004)

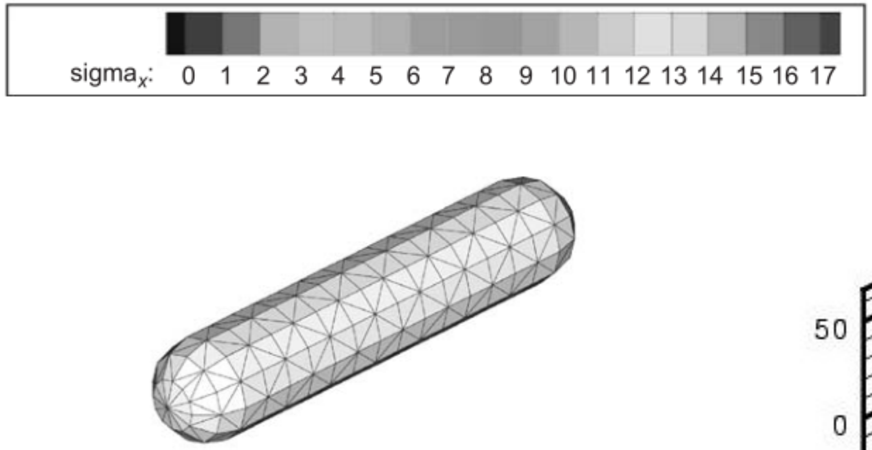
Fuel Cells

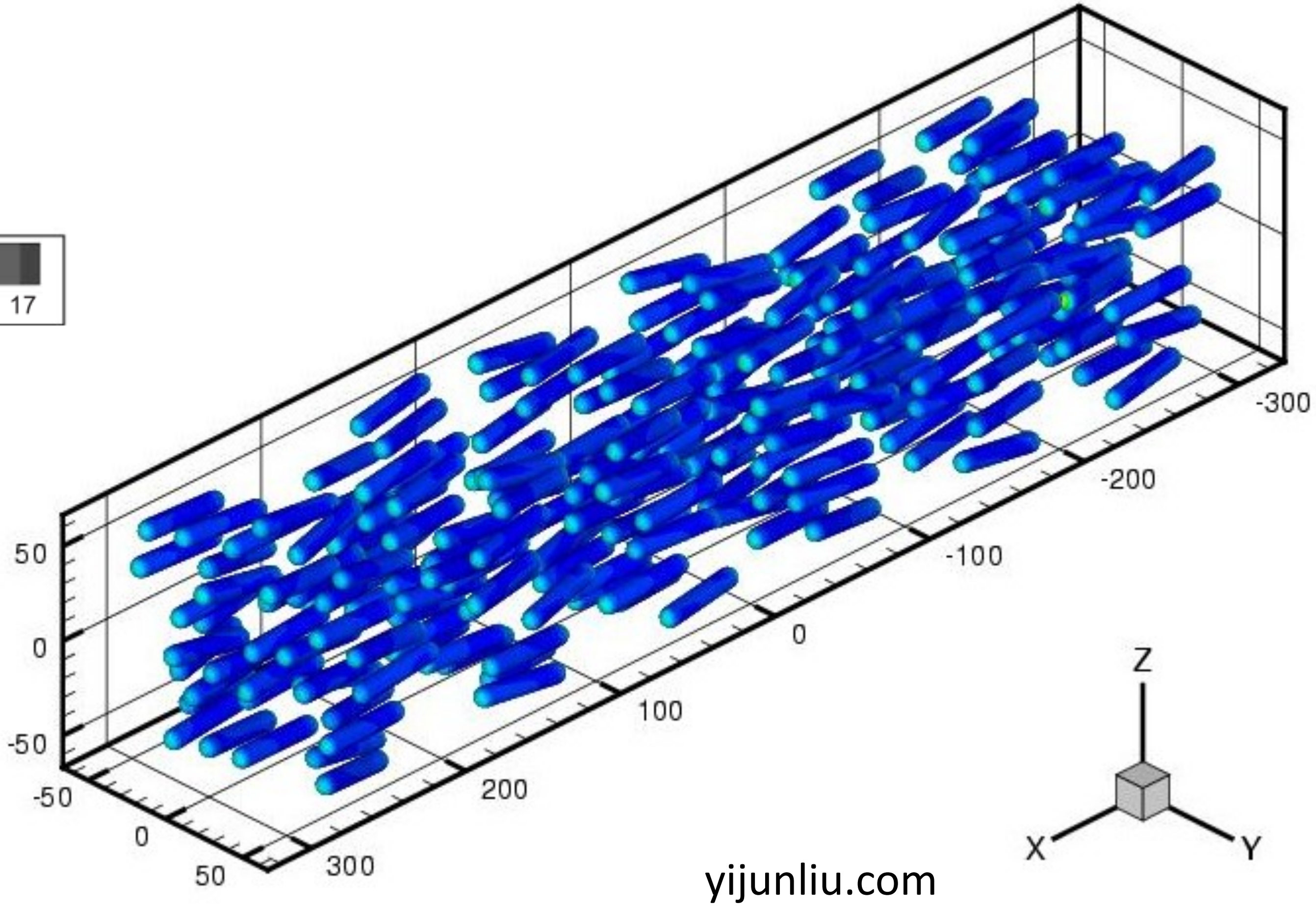


yijunliu.com

7. Example Applications

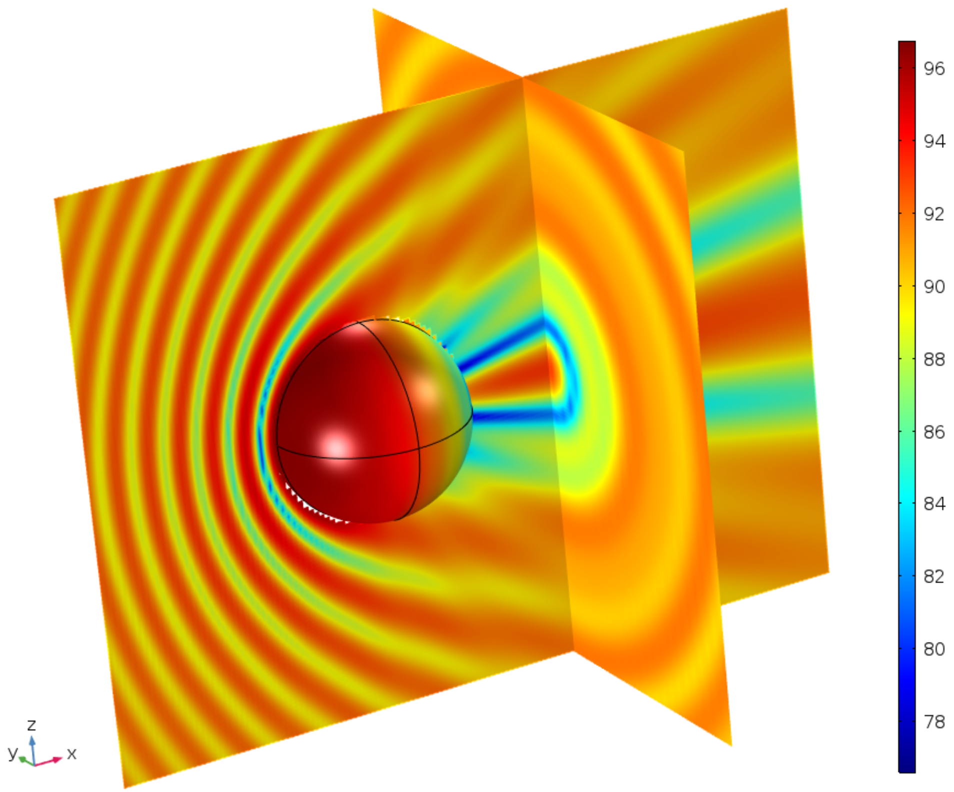
Elasticity: Fiber Composites

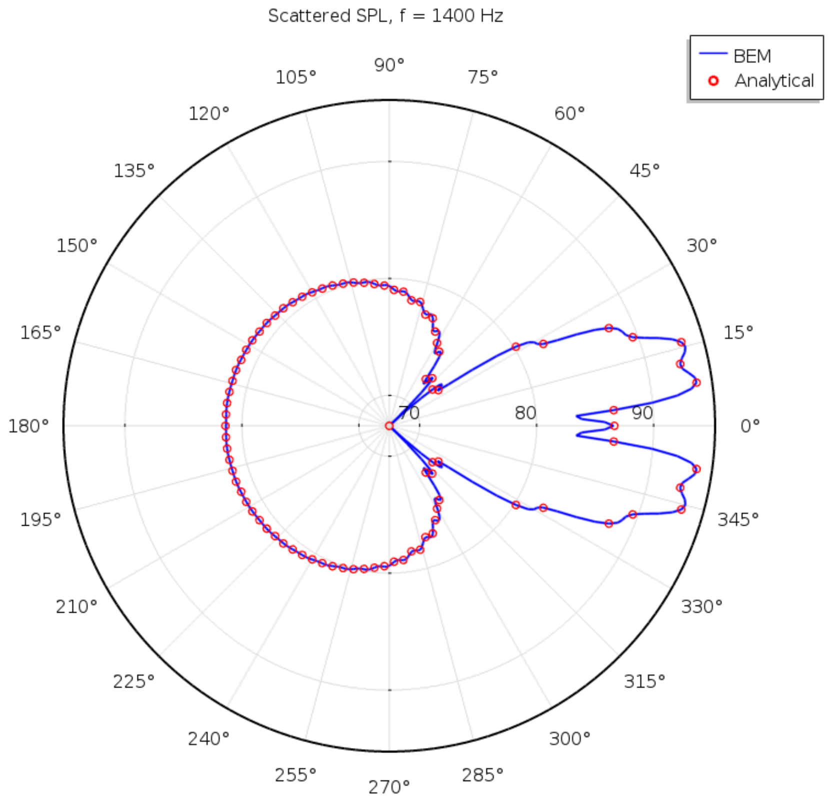




7. Example Applications

Acoustics: Sound pressure level

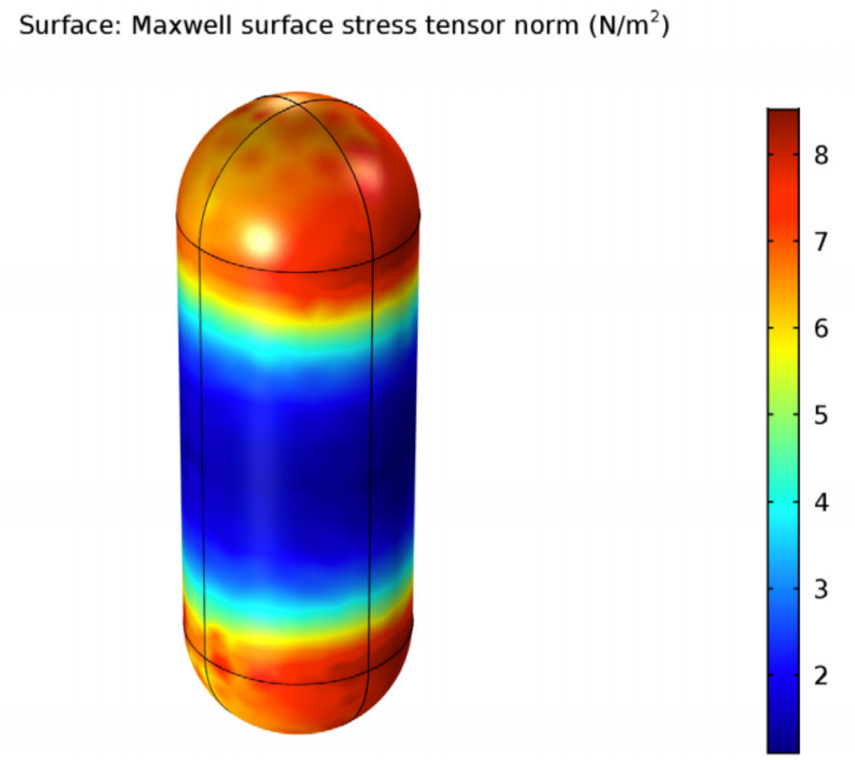


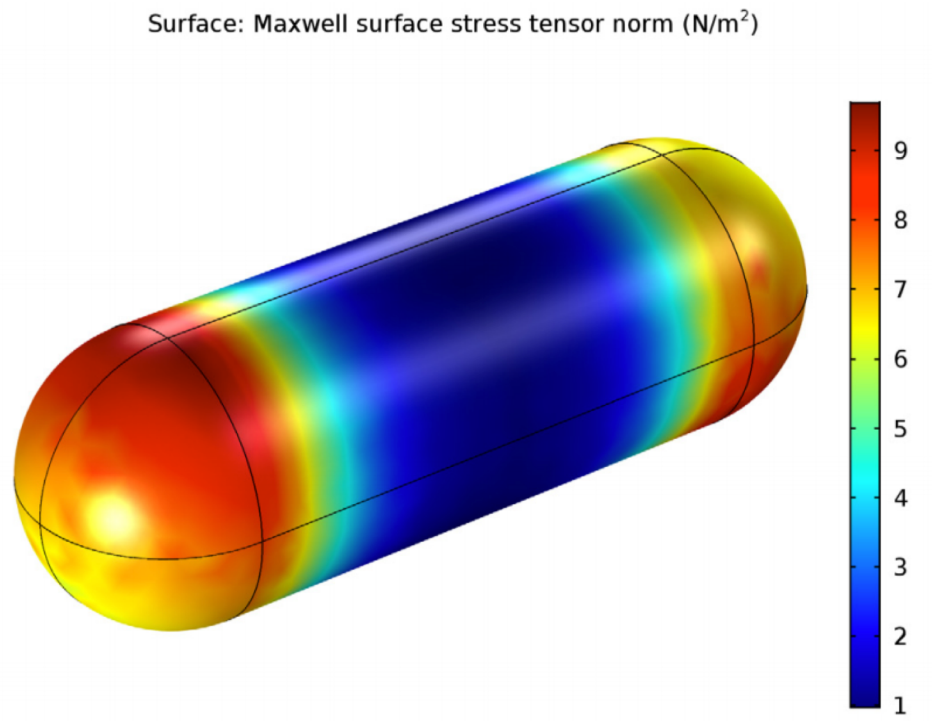


[*Spherical Scatterer: BEM Benchmark*]

7. Example Applications

Magnetostatics Modeling





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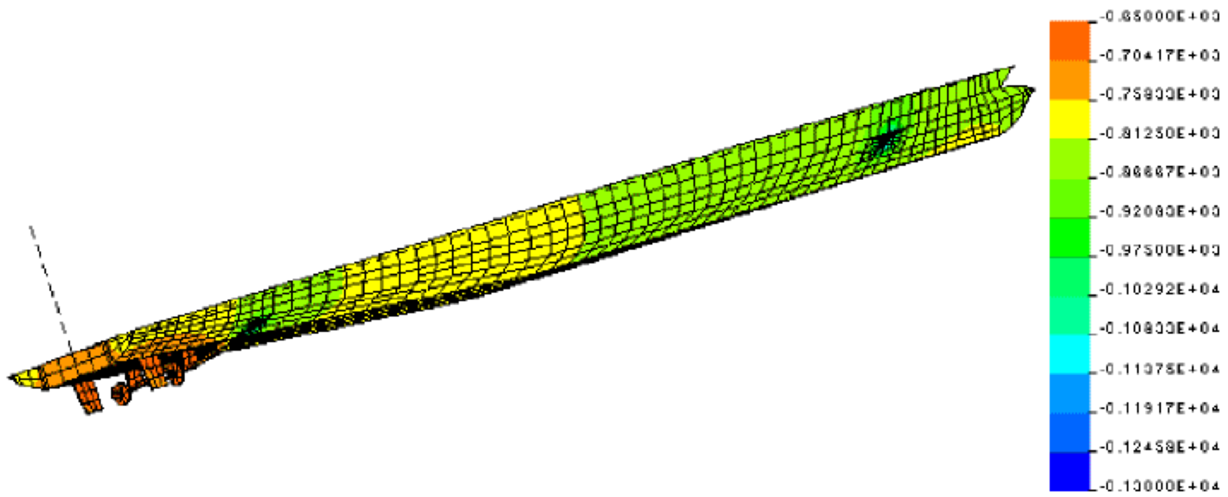
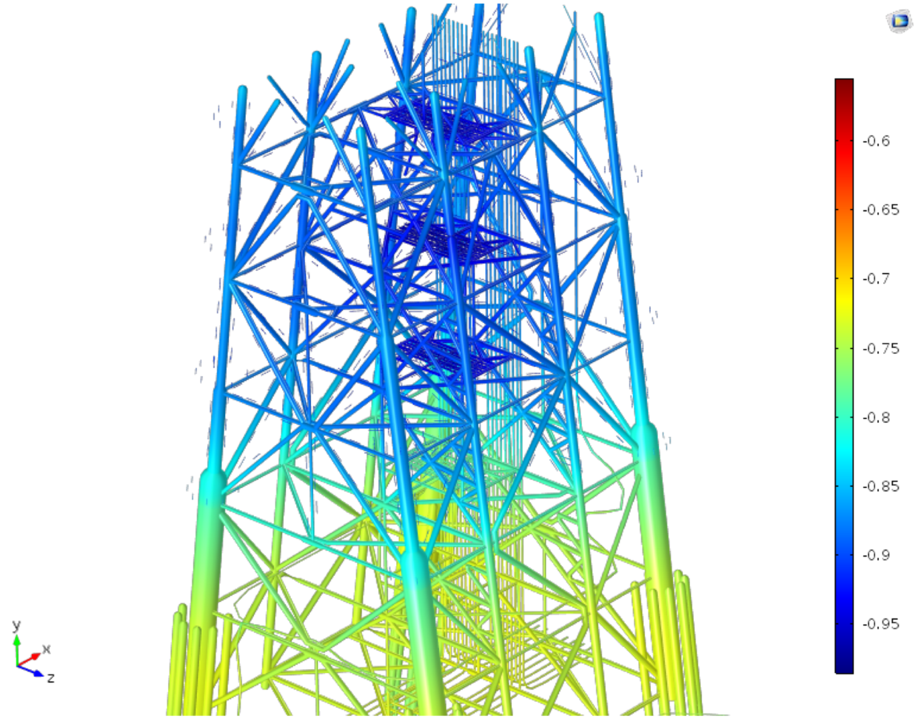
7. Example Applications

Corrosion rate: ship



corrosion-doctors.org

Jacket structure submerged in seawater



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