Boundary Element Method (BEM)

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1. Introduction

Nearly all physical phenomena occurring in nature can be describe by DEs and BCs.

**Analytical Solution**
- Can obtained for few specific problems with simple BCs
- Satisfy both DE and BCs

**Numerical Solution**
- Applicable for realistic scenario of practical engineering problems (approximate solutions)
- Satisfy one of the two (DE or BCs) and minimize the error in satisfying the other one.
- **Boundary Element Method (BEM)**: satisfy the DE exactly and minimize error in the satisfaction of BCs.
1. Introduction: The Idea of BEM

• Foundation idea of BEM came from Trefftz (1926), that we can approximate the solution to a PDE by looking at the solution to the PDE on the boundary and then use that information to find the solution inside the domain.

• As a consequence, the number of discretized elements is way less than FEM or FDM.

• BEM can be applied for potential problems governed by a DE that satisfied the Laplace equation or behaviors that has relating fundamental solutions: fluid flow, torsion of bars, diffusion and steady state heat conduction...

• Also useful for problems with complicated geometries, infinite domain problems.
1. Introduction: Advantages of BEM

Reduction of problem dimension by 1.

- Less data preparation time.
- Easier to change the applied mesh.

Uses less number of nodes and elements.

- Faster compute time and less storage.

No approximations imposed on the solution at interior point.

- High accuracy.
- Able to modeling problems of rapidly changing stresses.

Internal points of the domain are optional.

- Filter out unwanted information, focus on section of the domain of interested.
- Further reduces compute time.
1. Introduction: Disadvantages of BEM

- For non-linear problems, the interior must be modelled, especially in non-linear material problems.

- Poor for thin structures 3-D analysis, due to large surface/volume ratio and the close proximity of nodal points on either side of the structure thickness. Causing inaccuracies in the numerical integrations.

- Requires explicit knowledge of a fundamental solution of the PDE.

- The solution matrix resulting from the BE formulation is unsymmetric and fully populated with non-zero coefficients, this means that the entire BE solution matrix must be saved in the computer core memory.
**FEM vs. BEM**

- Discretization of whole domain
- Good on finite domains
- Approximates interior point solution \( u \) & BCs solution \( q \) must be found from \( u \) and approximation of \( q \) may not be as accurate
- Requires no prior knowledge of solution
- Solves most linear second-order PDEs

- Discretization of boundary
- Good on infinite or semi-infinite domains
- Approximates BCs solution \( q \) & interior point solution \( u \) approximation of \( q \) is accurate
- Requires knowledge of PDE solution
- Can be difficult to solve inhomogeneous or nonlinear problems

Fedele F. et al. (2005)
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2. Historical Perspective

- **C. F. Gauss (1813)**
  - Developed the Divergence Theorem.

- **G. Green (1828)**
  - Wrote a famous essay on the application of mathematical analysis to the theories of electricity and magnetism.
2. Historical Perspective

- E. I. Fredholm (1903)
  - Proved the existence and uniqueness of solution of the linear integral equation.

- M. A. Jaswon and A. R. Ponter (1963)
  - First formulated 2D potential problem in terms of a direct Boundary Integral Equation (BIE) and solved it numerically.
2. Historical Perspective

• F. J. Rizzo (1967)
  • Extended the work into the 2D elastostatic case.

• T. A. Cruse and F. J. Rizzo (1968)
  • Extended the work into 2D elastodynamics case.
2. Historical Perspective

• P. K. Banerjee and R. Butterfield (1975)
  • Coined the term “Boundary Element Method” in an attempt to make an analogy with Finite Element Method (FEM).

• C. A. Brebbia (1978)
  • Published the first textbook on BEM, ‘The boundary Element Method for Engineers’.

• From late 1970s, the number of journal articles shows an exponential grow rate.
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1. Unitary impulse function

\[ \delta(x) = \begin{cases} \frac{1}{2a}, & \text{if } -a < x < a \\ 0, & \text{if } x < -a \text{ or } x > a \end{cases} \]
1. Unitary impulse function

\[ \delta(x) = \begin{cases} 
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0, & \text{if } x < -a \text{ or } x > a 
\end{cases} \]

As we make “a” tends to zero from both sides, the function goes to infinity (called **instantaneous** impulse)

1. Unitary impulse function

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From our definition, it is possible to note that:

\[
\int_{-\infty}^{+\infty} \delta(x) \, dx = 1
\]

The integral of such function is zero anywhere but at the location of the origin, where it is equivalent to 1.

\[
\delta(x) = \begin{cases} 
\frac{1}{2a}, & \text{if } -a < x < a \\
0, & \text{if } x \leq -a \text{ or } x \geq a
\end{cases}
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What if impulse is not located at the origin?

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only when \( x = \xi \! \)

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only when \( x = \xi \)

Now we are able to understand the sifting property:

\[
\int_{-\infty}^{+\infty} \delta(\xi - x)F(\xi) \, d\xi = F(x)
\]

Attention: we are changing the domain of integration from \( x \) to \( \xi \)

2. Green’s function

Let’s us explore the diffusion-type equation (linear and inhomogeneous):

\[
\frac{\partial \phi}{\partial t} - D \nabla^2 \phi = F
\]

**D** – diffusivity constant

\( \phi \) – any diffusing physical quantity function of space and time

\( F \) – forcing term (source or sink) function of space and time

**OBS:** The IC and BC’s imposed will depend on the physical system being described

2. Green’s function

Let’s us explore the diffusion-type equation (linear and inhomogeneous):

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\]

D – diffusivity constant
\phi – any diffusing physical quantity function of space and time
F – forcing term (source or sink) function of space and time

OBS: The IC and BC’s imposed will depend on the physical system being described

Sometimes it is easier to find the solution to our problem by invoking an auxiliary problem:

\[
L(G) = \delta(x - \xi)\delta(t - \tau)
\]

The same differential operator but now we want are seeking for the system response when subject to an instantaneous forcing term!

2. Green’s function

Typically, we call the “Free Space Green’s Function” the solution that only satisfies the differential operator

When working with the diffusion-type equation, the Green’s function is just a Gaussian function:

\[
G(x, t; \xi, \tau) = \frac{1}{D \sqrt{4\pi(t - \tau)}} \exp \left[ -\frac{(x - \xi)^2}{4D^2(t - \tau)} \right]
\]

Caution: function is not well defined at (\xi, \tau)

2. Green’s second Identity

Given the two solutions \( G \) and \( \phi \):

\[
\iiint_V (G \nabla^2 \phi - \phi \nabla^2 G) dV = \iint_{\partial V} \left( G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n} \right) ds
\]

Reducing one dimension
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4. Governing Equations

General Mass Balance Equation

\[-\nabla \cdot (\rho \vec{v}) = \frac{\partial (\rho \phi)}{\partial t}\]
4. Governing Equations

General Mass Balance Equation

\[-\nabla \cdot (\rho \vec{v}) = \frac{\partial (\rho \phi)}{\partial t}\]

Our conventional fluid mechanics classes

\[\phi = 1\]

\[-\nabla \cdot (\rho \vec{v}) = \frac{\partial \rho}{\partial t}\]

\[\rho \nabla \cdot \vec{v} = 0\]

\[\nabla \cdot \vec{v} = 0\]
4. Governing Equations

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Our conventional fluid mechanics classes

\[\phi = 1\]

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\[\rho \nabla \cdot \vec{v} = 0\]

\[\nabla \cdot \vec{v} = 0\]

Fluid flow through porous media

We need a “bridge” that will help us to make our equation suitable to porous media flow (constitutive equation)

\[\vec{v} = -\frac{k}{\mu} \nabla P\]
4. Governing Equations

Fluid flow through porous media

\[
\vec{v} = -\frac{k}{\mu} \nabla P
\]

Darcy's Law

\[
-\nabla \cdot (\rho \vec{v}) = \frac{\partial (\rho \phi)}{\partial t}
\]

\[
\nabla \cdot \left( \frac{k}{\mu} \rho \nabla P \right) = \frac{\partial (\rho \phi)}{\partial t}
\]
4. Governing Equations

Fluid flow through porous media

\[ \vec{v} = -\frac{k}{\mu} \nabla P \]

\[ -\nabla \cdot (\rho \vec{v}) = \frac{\partial (\rho \phi)}{\partial t} \]

\[ \nabla \cdot \left( \frac{k}{\mu} \rho \nabla P \right) = \frac{\partial (\rho \phi)}{\partial t} \]

... working with the RHS ...

\[ \nabla \cdot \left( \frac{k}{\mu} \rho \nabla P \right) = \rho \phi c_t \frac{\partial P}{\partial t} \]
4. Governing Equations

Fluid flow through porous media

\[ \vec{v} = -\frac{k}{\mu} \nabla P \]

Darcy’s Law

\[ -\nabla \cdot (\rho \vec{v}) = \frac{\partial (\rho \phi)}{\partial t} \]

\[ \nabla \cdot \left( \frac{k}{\mu} \rho \nabla P \right) = \frac{\partial (\rho \phi)}{\partial t} \]

Governing equation

\[ \nabla^2 P = \frac{\mu \phi c_t}{k} \frac{\partial P}{\partial t} + Q \]

Sink term (producing well)

Homogeneous and isotropic media

Slightly-compressible fluid
4. Governing Equations

Integral solution protocol – the basis of Boundary Element Method

\[ \nabla^2 P = \frac{\mu \phi c_t}{k} \frac{\partial P}{\partial t} + Q \]

Writing in terms of the differential operator “L”

\[ L \equiv \nabla^2 \left( \frac{\partial}{\partial t_{DA}} \right) \Rightarrow L(P_D) = Q_D \]

By recognizing the fundamental solution of L as “G”:

\[ L(G) \equiv \nabla^2 G - \frac{\partial G}{\partial t_{DA}} = \delta(X_D - \xi)\delta(t_{DA} - \tau) \]
4. Governing Equations

\[ \nabla^2 P_D - \frac{\partial P_D}{\partial t_{DA}} = Q_D \]

- George Green (1793 – 1841)
  - Fundamental solution (Green’s function)
  - Green’s second identity

- Johann C. F. Gauss (1777 – 1855)
  - Divergence (Gauss) theorem

\[ P_D(\vec{X}_D, t_{DA}) = \int_0^{t_{DA}} \int_{\partial D} \left[ G \frac{\partial P_D}{\partial n} - P_D \frac{\partial G}{\partial n} \right] ds d\tau - \int_0^{t_{DA}} \int_{\partial D} G Q_D d\xi d\tau \]

Valid within the domain “D”

Usually either \( P_D \) or \( \frac{\partial P_D}{\partial n} \) are specified as BC’s

Boundary Element Method Protocol
4. Governing Equations

**Boundary Element Method**

Every point within the domain (including the boundary) could be represented by a boundary integral solution with the Green function (differential equation is linear).

In general form the boundary integral solution can be represented by:

\[
\beta(\xi, \eta)\phi(x, y) = \oint_{\partial \Omega} \left[ \frac{\partial G(x, y; \xi, \eta)}{\partial n} \phi(x, y) - G(x, y; \xi, \eta) \frac{\partial \phi(x, y)}{\partial n} \right] ds
\]

\[
\beta = \begin{cases} 
1, & \text{if } (\xi, \eta) \text{ is inside domain } \Omega \\
\frac{1}{2}, & \text{if } (\xi, \eta) \text{ is on smooth boundary } \partial \Omega
\end{cases}
\]
4. Governing Equations

The value of random point requires 2 boundary conditions from each boundary element;
N boundary elements, each boundary elements has 2 boundary conditions (one is prescribed, and one is unknown);
N unknowns require N equations!
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Imagine inside a fluid flow field and a square domain exists. 2 boundaries are specified with flow potential and 2 boundaries are specified with the \( \frac{\partial \phi}{\partial n} \) (which could be considered as velocity given). What is the other B.C. on each boundary?

4 boundaries; 4 B.C. given
5. Hand-Calculation Example

\[
\int_\partial \left[ \left( \frac{\partial G(x, y; \xi, \eta)}{\partial n} - \beta(\xi, \eta) \right) \phi(x, y) \right] ds(x, y) = \int_\partial \left[ G(x, y; \xi, \eta) \frac{\partial \phi(x, y)}{\partial n} \right] ds(x, y)
\]

If

\[
G(x, y; \xi, \eta) = \frac{1}{2\pi} \left( x(\ln(r) - x + y\theta) \right) \bigg|_1^2
\]

\[
V_x = \frac{\partial G}{\partial x} = -\frac{k}{2\pi} \left( \ln(r) \right) \bigg|_1^2
\]

\[
V_y = \frac{\partial G}{\partial y} = -\frac{k}{2\pi} \left( \ln(\theta) \right) \bigg|_1^2
\]

\[
r = \sqrt{(x - \xi)^2 + (y - \eta)^2}
\]

\[
k = \text{const} = 10
\]

\[
G_{11}(0, 0.5; 0, 0.5) = \frac{1}{2\pi} \left( 0.5(\ln(0.5) - 0.5) - (-0.5(\ln(0.5) + 0.5)) \right)
\]

\[
= -0.27
\]
5. Hand-Calculation Example

\[
\oint \left[ \left( \frac{\partial G(x, y; \xi, \eta)}{\partial n} - \beta(\xi, \eta) \right) \phi(x, y) \right] ds(x, y) = \oint \left[ G(x, y; \xi, \eta) \frac{\partial \phi(x, y)}{\partial n} \right] ds(x, y)
\]

If

\[
G(x, y; \xi, \eta) = \frac{1}{2\pi} (x \ln(r) - x + y\theta) \bigg|_1^2
\]

\[
V_x = \frac{\partial G}{\partial x} = -\frac{k}{2\pi} \ln(r) \bigg|_1^2
\]

\[
V_y = \frac{\partial G}{\partial y} = -\frac{k}{2\pi} \ln(\theta) \bigg|_1^2
\]

\[
r = \sqrt{(x - \xi)^2 + (y - \eta)^2}
\]

\[
k = \text{const} = 10
\]

\[
G_{31}(0, -0.5; 0, 0.5)
\]

\[
= \frac{1}{2\pi} \left[ 0.5 \left( \ln \left( \frac{\sqrt{5}}{2} \right) - 0.5 - 1.117 - (-0.5 \ln \left( \frac{\sqrt{5}}{2} \right) + 0.5 - 2.034) \right) \right] = 0.0062
\]
5. Hand-Calculation Example

\[ \oint_{\partial V} \left[ \left( \frac{\partial G(x, y; \xi, \eta)}{\partial n} - \beta(\xi, \eta) \right) \phi(x, y) \right] ds(x, y) = \oint_{\partial V} \left[ G(x, y; \xi, \eta) \frac{\partial \phi(x, y)}{\partial n} \right] ds(x, y) \]

Given:

\[ G(x, y; \xi, \eta) = \frac{1}{2\pi} (x (\ln(r) - x + y\theta))^2 \]

If \( k = \text{const} = 10 \)

\[ r = \sqrt{(x - \xi)^2 + (y - \eta)^2} \]

\[ \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} = \begin{bmatrix} -0.27 & -0.053 & 0.006 & -0.053 \\ -0.053 & -0.27 & -0.053 & 0.006 \\ 0.006 & -0.053 & -0.27 & -0.053 \\ -0.053 & 0.006 & -0.053 & -0.27 \end{bmatrix} \]
5. Hand-Calculation Example

\[
\int_{\partial V} \left[ \left( \frac{\partial G(x,y;\xi,\eta)}{\partial n} - \beta(\xi,\eta) \right) \phi(x,y) \right] ds(x,y) = \int_{\partial V} \left[ G(x,y;\xi,\eta) \frac{\partial \phi(x,y)}{\partial n} \right] ds(x,y)
\]

\[
G(x,y;\xi,\eta) = \frac{1}{2\pi} \left( x(\ln(r) - x + y\theta) \right)^2
\]

\[
V_x = \left. \frac{\partial G}{\partial x} \right|_1 = -\frac{k}{2\pi} (\ln(r))^2
\]

\[
V_y = \left. \frac{\partial G}{\partial y} \right|_1 = -\frac{k}{2\pi} (\ln\theta)^2
\]

\[
r = \sqrt{(x - \xi)^2 + (y - \eta)^2}
\]

\[
k = \text{const} = 10
\]

\[
V_{11}(0,0.5; 0,0.5) = \frac{10}{2\pi} \left( \frac{\pi}{2} - \frac{3}{2\pi} \right) = 5.00
\]
5. Hand-Calculation Example

\[ \oint_{\partial V} \left[ \left( \frac{\partial G(x, y; \xi, \eta)}{\partial n} - \beta(\xi, \eta) \right) \phi(x, y) \right] ds(x, y) = \oint_{\partial V} \left[ G(x, y; \xi, \eta) \frac{\partial \phi(x, y)}{\partial n} \right] ds(x, y) \]

\[
\begin{align*}
G(x, y; \xi, \eta) & = \frac{1}{2\pi} \left( x \ln(r) - x + y\theta \right) \\
V_x & = \frac{\partial G}{\partial x} = -\frac{k}{2\pi} (\ln(r)) \\
V_y & = \frac{\partial G}{\partial y} = -\frac{k}{2\pi} (\ln\theta) \\
r & = \sqrt{(x - \xi)^2 + (y - \eta)^2} \\
k & = \text{const} = 10
\end{align*}
\]

\[ V_{12}(0, -0.5; 0, 0.5) = \frac{10}{2\pi} \left[ \ln 0.5 - \ln \left( \frac{\sqrt{5}}{2} \right) \right] = -1.281 \]
5. Hand-Calculation Example

\[
\int_{\Gamma} \left[ \frac{\partial G(x, y; \xi, \eta)}{\partial n} - \beta(\xi, \eta) \right] \phi(x, y) \, ds(x, y) = \int_{\Gamma} \left[ G(x, y; \xi, \eta) \frac{\partial \phi(x, y)}{\partial n} \right] \, ds(x, y)
\]

\[
G(x, y; \xi, \eta) = \frac{1}{2\pi} (x(\ln(r) - x + y\theta)) \bigg|_{1}^{2}
\]

\[
V = \frac{\partial G(x, y; \xi, \eta)}{\partial n} = -\frac{k}{2\pi} (\ln(r)) \bigg|_{1}^{2}
\]

\[
r = \sqrt{(x - \xi)^2 + (y - \eta)^2}
\]

\[
k = \text{const} = 10
\]

\[
\begin{bmatrix}
V_{11} & V_{12} & V_{13} & V_{14} \\
V_{21} & V_{22} & V_{23} & V_{24} \\
V_{31} & V_{32} & V_{33} & V_{34} \\
V_{41} & V_{42} & V_{43} & V_{44}
\end{bmatrix}
= \begin{bmatrix}
5.000 & -1.281 & -1.476 & -1.281 \\
-1.281 & 5.000 & -1.281 & -1.476 \\
-1.476 & -1.281 & 5.000 & -1.281 \\
-1.281 & -1.476 & -1.281 & 5.000
\end{bmatrix}
\]
5. Hand-Calculation Example

\[
\begin{bmatrix}
V_{11} - \beta & V_{12} & V_{13} & V_{14} \\
V_{21} & V_{22} - \beta & V_{23} & V_{24} \\
V_{31} & V_{32} & V_{33} - \beta & V_{34} \\
V_{41} & V_{42} & V_{43} & V_{44} - \beta
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4
\end{bmatrix}
= 
\begin{bmatrix}
G_{11} & G_{12} & G_{13} & G_{14} \\
G_{21} & G_{22} & G_{23} & G_{24} \\
G_{31} & G_{32} & G_{33} & G_{34} \\
G_{41} & G_{42} & G_{43} & G_{44}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \phi}{\partial n}_1 \\
\frac{\partial \phi}{\partial n}_2 \\
\frac{\partial \phi}{\partial n}_3 \\
\frac{\partial \phi}{\partial n}_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
\phi_1 \\
\frac{\partial \phi}{\partial n}_2 \\
\phi_3 \\
\frac{\partial \phi}{\partial n}_4
\end{bmatrix}
= 
\begin{bmatrix}
0.409 \\
-16.284 \\
0.409 \\
8.987
\end{bmatrix}
\]
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6. Numerical Example

- The general boundary integral solution

\[ \lambda(\xi, \eta) \overline{p_D}(\xi, \eta; s) = \int_{\partial \Omega} \left( \overline{G} \frac{\partial \overline{p_D}}{\partial n} - \overline{p_D} \frac{\partial \overline{G}}{\partial n} \right) dS + \int_{\Omega} \overline{G} \overline{Q_D} d\omega \]

Reservoir and Fluid Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reservoir dimensions x,y</td>
<td>200ft</td>
</tr>
<tr>
<td>Initial pressure ( p_i )</td>
<td>5000psia</td>
</tr>
<tr>
<td>Thickness ( h )</td>
<td>50ft</td>
</tr>
<tr>
<td>Permeability ( k )</td>
<td>0.01md</td>
</tr>
<tr>
<td>Pore Compressibility ( c_p )</td>
<td>( 10^{-12} ) 1/psi</td>
</tr>
<tr>
<td>Porosity ( \phi )</td>
<td>0.3</td>
</tr>
<tr>
<td>Fluid viscosity ( \mu )</td>
<td>0.32cp</td>
</tr>
<tr>
<td>Fluid compressibility ( c_f )</td>
<td>( 3 \times 10^{-6} ) 1/psi</td>
</tr>
<tr>
<td>Fracture half length ( x_f )</td>
<td>50ft</td>
</tr>
<tr>
<td>Bottomhole pressure ( p_{wf} )</td>
<td>1000psia</td>
</tr>
<tr>
<td>Fracture Conductivity ( k_f )</td>
<td>Case A: 5 md-ft, Case B: 10 md-ft, Case C: 50 md-ft</td>
</tr>
</tbody>
</table>

Mesh free!

Oil single phase;
BEM-FVM combined method!

6. Numerical Example

Application: Slightly-compressible fluid flow (Oil) with fractures

\[ \nabla^2 p_D = \frac{\partial p_D}{\partial t_D} - Q_D(t_D) \]

Laplace Transform

\[ (\nabla^2 - s)p_D + Q_D = 0 \]

Multiply by the Green’s function and integrate over the space domain

\[ \int_{\Omega} \left[ G(\nabla^2 - s)p_D - GQ_D \right] d\omega = 0 \]

Expanding

\[ - \int_{\Omega} \nabla \cdot \left( G \nabla p_D - p_D \nabla G \right) d\omega + \int_{\Omega} p_D \left( \nabla^2 G - s G \right) d\omega - \int_{\Omega} G Q_D d\omega = 0 \]
Application: Slightly-compressible fluid flow (Oil) with fractures

\[ = - \int_{\Omega} \nabla \cdot \left( \overrightarrow{G} \nabla \overline{p}_D - \overline{p}_D \nabla \overrightarrow{G} \right) d\omega + \int_{\Omega} \overline{p}_D (\nabla^2 \overrightarrow{G} - s \overrightarrow{G}) d\omega - \int_{\Omega} \overrightarrow{G} \overline{Q}_D d\omega = 0 \]

First term:

\[ = - \int_{\Omega} \nabla \cdot \left( \overrightarrow{G} \nabla \overline{p}_D - \overline{p}_D \nabla \overrightarrow{G} \right) d\omega \]

\[ = - \int_{\partial\Omega} \left( \overrightarrow{G} \nabla \overline{p}_D - \overline{p}_D \nabla \overrightarrow{G} \right) \cdot \mathbf{n} dS \]

Second term:

\[ \int_{\Omega} \overline{p}_D (\nabla^2 \overrightarrow{G} - s \overrightarrow{G}) d\omega \]

\[ = - \int_{\Omega} \overline{p}_D \delta(x_D - x_{D_0}) \delta(y_D - y_{D_0}) d\omega = -\overline{p}_D(\xi; \eta; s) \]

Boundary integral solution in Laplace domain:

\[ \beta(\xi, \eta) \overline{p}_D(\xi, \eta; s) = \int_{\partial\Omega} \left( \frac{\partial}{\partial \mathbf{n}} \overline{G} \overline{p}_D - \overline{p}_D \frac{\partial \overrightarrow{G}}{\partial \mathbf{n}} \right) dS + \int_{\Omega} \overrightarrow{G} \overline{Q}_D d\omega \]

\[ \beta(\xi, \eta) = \begin{cases} 1, & \text{if } (\xi, \eta) \text{ is inside domain } \Omega \\
\frac{1}{2}, & \text{if } (\xi, \eta) \text{ is on smooth boundary } \partial\Omega \end{cases} \]
6. Numerical Example

- Efficient and implicit;
- No iteration is required!

Flow chart for oil single phase using Boundary Element Method
**Boundary Element Method (BEM)**

### Numerical Example

The general boundary integral solution

\[
\mathbf{u} = \mathbf{G} \cdot \mathbf{f} + \mathbf{K} \cdot \mathbf{u}_0
\]

where \( \mathbf{G} \) and \( \mathbf{K} \) are the Green's function and the stiffness matrix, respectively, and \( \mathbf{f} \) and \( \mathbf{u}_0 \) are the load and initial displacement vectors. The solution can be computed using a mesh-free approach.

---

**Reference:**

6. Numerical Example

**Reservoir and Fluid Properties**

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reservoir dimensions x,y</td>
<td>200ft</td>
</tr>
<tr>
<td>Initial pressure $p_i$</td>
<td>5000 psia</td>
</tr>
<tr>
<td>Thickness $h$</td>
<td>50ft</td>
</tr>
<tr>
<td>Permeability $k$</td>
<td>0.01 md</td>
</tr>
<tr>
<td>Pore Compressibility $c_\phi$</td>
<td>$10^{-12}$ l/psi</td>
</tr>
<tr>
<td>Porosity $\phi$</td>
<td>0.3</td>
</tr>
<tr>
<td>Fluid viscosity $\mu$</td>
<td>0.32 cp</td>
</tr>
<tr>
<td>Fluid compressibility $c_f$</td>
<td>$3 \times 10^{-6}$ l/psi</td>
</tr>
<tr>
<td>Fracture half length $x_f$</td>
<td>50ft</td>
</tr>
<tr>
<td>Bottomhole pressure $p_{wf}$</td>
<td>1000 psia</td>
</tr>
</tbody>
</table>

**Fracture Conductivity**

<table>
<thead>
<tr>
<th>Case</th>
<th>$k_f w_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5 md-ft</td>
</tr>
<tr>
<td>B</td>
<td>10 md-ft</td>
</tr>
<tr>
<td>C</td>
<td>50 md-ft</td>
</tr>
</tbody>
</table>

- Good agreement between results with BEM and in-house FVM simulators’!
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6. Numerical Example
7. Example Applications
7. Example Applications

Thermal Analysis

Fuel Cells

Sutradhar et al. (2004)

yijunliu.com
7. Example Applications

Elasticity: Fiber Composites

![Diagram of fiber composites with stress distribution]

\[ \sigma_{x}: 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17 \]
7. Example Applications

Acoustics: Sound pressure level

_Spherical Scatterer: BEM Benchmark_
7. Example Applications

Magnetostatics Modeling

Surface: Maxwell surface stress tensor norm (N/m²)
7. Example Applications

Corrosion rate: ship

Jacket structure submerged in seawater

corrosion-doctors.org

comsol.com
References

References


References


References

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- http://corrosion-doctors.org/