MULTIPHASE FLOW
GeoEE 500

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MULTIPHASE FLUIDS

- Twin or multiple fluid phases
  - Reactors - Bubbles
  - Spraying
  - Porous media - Oil/water/gas + Solid

- Similar processes apply in each.
- Have additional complications if the presence of interfacial tension in contributing to response, e.g.

- Extra non-dimensional parameter.
  - e.g. Stokes settling: $Re \rightarrow Friction factor$ $Fr \rightarrow$ Buoyant motion

$Wob No. = \frac{Inertial force}{Surface tension}$
In the absence of porous medium — Bubble Mechanics

\[ \text{Region 1: } \frac{\text{Re}}{0.3} \]
\[ \text{Region 2: } \frac{\text{Re}}{7.4} \]
\[ \text{Region 3: } \frac{\text{Re}}{8} \]
\[ \text{Region 4: } \frac{\text{Re}}{1880} \]

Controlled by 3 parameters:

Primarily by parameters of the fluid (not gas): \( \rho, \mu, \sigma \) + \( g + D + V \)

\[ \text{Re} = \frac{D\sqrt{\rho}}{\mu} = \text{Inertial force} \]

\[ \text{Fr} = \frac{V^2}{Dg} = \frac{\rho e}{\rho e - \rho g} \]

\[ \text{We} = \frac{Dv^2\rho}{\sigma} = \text{Inertial force} \]

Note: Also the Bond No in porous media flow

\[ \text{Bond No} = B_o = \frac{\text{Gravity}}{\text{Surface tension}} = \frac{\text{We}}{\text{Fr}} = \frac{\rho D^2 g}{\sigma} \]
If liquid defined a priori, then steady rise of bubble is constrained in the plane defined by:

\[ \text{Re}^4 \text{Fr} \text{ We}^{-3} = \frac{\rho S^2 g}{\mu^2} = \text{Constant} = C \]

Bubbles must propagate in this plane, since the properties define a single fluid.

\[ 4 \log \text{Re} + \log \text{Fr} + 3 \log \left( \frac{1}{\text{We}} \right) = \log C = \text{constant} \]

May define behavior of 2 fluids in this plane.

1. Define C for any fluid at any Temp Fig 10.11.3
2. Behavior confined to this plane e.g. Bubble shape = \( f(\text{Re}) \) only Fig. 10.11.4
3. Bubble trajectories shown Fig 10.11.5
4. Evaluate Drag Coeffs.

<table>
<thead>
<tr>
<th>Region</th>
<th>Drag Coeff, ( C_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24 ( \text{Re}^{-1} )</td>
</tr>
<tr>
<td>2</td>
<td>18.2 ( \text{Re}^{-0.682} )</td>
</tr>
<tr>
<td>3</td>
<td>0.366 ( \text{WeFr}^{-1} )</td>
</tr>
<tr>
<td>4</td>
<td>2.61</td>
</tr>
</tbody>
</table>
Fig. 10.11.3. Liquid constant $C$ for various liquids as a function of temperature.

Fig. 10.11.4. Spatial representation of the liquid plane with the existence ranges of the different forms of bubbles.

Fig. 10.11.5. Representation of various flow processes in the liquid plane for $C = 3.9 \times 10^6$ (e.g. water at 20°C). — Rate of rise of bubbles; curve ABCD after Peebles and Garber (1953); curve DE after Davies and Taylor (see p. 705, footnote †). EJH Results of Van Krevelen and Hoftijser (1950). — ABF Rate of rise of weightless sphere with $\zeta$ according to Fig. 10.3.1. — Stokes law taking Hadamard correction into account. — Curve according to W. M. Haberman and R. K. Morton for air bubbles in filtered and distilled water at 19°C. JK Falling film [eqns. (7.24.6) and (7.24.9)]. GH Surface waves [eqn. (10.11.6)]. For the literature references see p. 702, footnote †.

Fig. 10.12.1. The main shapes assumed by bubbles on rising in a liquid: (a) small spherical bubble with no internal circulation; (b) small spherical bubble with internal circulation; (c) elliptical bubble rising along a spiral path; (d) irregular-shaped mushroom bubble; (e) spherical-cap bubble, the upper boundary of which, according to Davies and Taylor, has exactly the shape of a spherical cap.
Gravitational Instability of the Bottom Boundary Layer

The initiation of plumes must clearly begin with the formation of upwelling regions. One approach is to investigate the gravitational instability of a horizontal layer of fluid lying under a denser layer of fluid. The lower layer will then develop Rayleigh-Taylor instability. The wavelength of maximum growth rate and the exponential time constant of growth have been theoretically and numerically predicted for a number of geometries and boundary conditions (Rayleigh 1900, Dobrin 1941, Chandrasekhar 1955, Danès 1964, Selig 1965, Ramberg 1963, 1967a, 1968, b.c, 1970, Biot 1966, Biot \\ & Ode 1965, Berner et al. 1972, Whitehead & Luther 1975, Marsh 1979, Whitehead et al. 1984). Demonstration experiments with putty and other non-Newtonian fluids have been extensively photographed and compared with geological formations by Nettleton (1934, 1943), Parker & McDowell (1955), and Ramberg (1963, 1967b, 1970). There was no intercomparison between the laboratory experiments and theory due to the unknown rheology of the laboratory materials.

The limit in which one layer is thin and of lower viscosity than the other is particularly relevant in the geophysical context. Take the configuration shown in Figure 6, in which a thin layer of density $\rho - \Delta \rho$, depth $d$, and viscosity $\mu_1$ lies under a semi-infinite fluid of density $\rho$ and viscosity $\mu_2$. Let the wavelength $l$ of the disturbance $\eta(x, t)$ to the horizontal interface be larger than the depth $d$ of the thin layer, so that the lateral velocity $u$ in the thin layer is larger than the vertical velocity. The scaling argument used here can be found in a number of the cited studies. The force balance is between the lateral pressure gradient $p/l$ and viscosity, and thus we have

$$\frac{p}{l} = \mu_1 \frac{u}{d}. \tag{1}$$

In the deep viscous fluid that lies overhead, the force balance is between stress due to the vertical velocity $w$ of the interface, pressure, and buoyancy, which gives us

$$\frac{p}{l} = \mu_2 \frac{w}{l} + \frac{g \Delta \rho \eta}{l}. \tag{2}$$

Combining (1) and (2), we have

$$\mu_1 \frac{u}{d} - \mu_2 \frac{w}{l} = \frac{g \Delta \rho \eta}{l}. \tag{3}$$

Using continuity

$$\frac{u}{l} + \frac{w}{d} = 0, \tag{4}$$

kinematics of the interface

$$\frac{\partial \eta}{\partial t} = w, \tag{5}$$

and the fact that the growth will be exponential ($w = we^\sigma$), we find that

$$\sigma = \frac{g \Delta \rho}{\mu_2} \frac{l}{\mu_1 \frac{d}{l} + 1}. \tag{6}$$

The growth rate $\sigma$ is shown as a function of the normalized wavelength $l/d$ in Figure 7. Note that the maximum growth occurs at

![Figure 6](image6.png)

**Figure 6** Definition parameters for the analysis of Rayleigh-Taylor instability of a thin layer lying under a deep fluid.

![Figure 7](image7.png)

**Figure 7** Growth rate as a function of wavelength for one layer undergoing Rayleigh-Taylor instability.
\[ \frac{l}{d} = \left( \frac{2\mu_2}{\mu_1} \right)^{1/3} = 1.26 \left( \frac{\mu_2}{\mu_1} \right)^{1/3}, \tag{7} \]

where the growth rate is
\[ \sigma = 0.42 \left( \frac{g\delta\rho_d}{\mu_2} \right)^{1/3} \left( \frac{\mu_2}{\mu_1} \right)^{1/3}. \tag{8} \]

The central finding here is that the fastest growth length scale is proportional to \((\mu_2/\mu_1)^{1/3}\) (which may be a large number). Two superior (but

\[ Figure 8 \ An \ experimental \ demonstration \ of \ Rayleigh-Taylor \ instability. \]
\[ T = \mu_2 \frac{w}{l} \]
\[ P \cdot l = T \cdot l + (g \Delta \rho) \eta \cdot l \]
\[ \therefore \frac{P}{l} = \frac{\mu_2 w}{l^2} + \frac{(g \Delta \rho) \eta}{l} \quad (2) \]

\[ P \cdot d = T \cdot l = \mu_1 \frac{u}{d} \cdot l \]
\[ \therefore \frac{P}{l} = \mu_1 \frac{u}{d^2} \quad (2) \]

\[ \frac{u \cdot d}{l \cdot d} + \frac{w \cdot l}{l \cdot d} = 0 \]
\[ \frac{u}{l} + \frac{w}{d} = 0 \quad (4) \]
IMMISCIBLE DISPLACEMENT - POROUS MEDIA

Controlled by properties of:

Fluids - \( \rho, \mu, \sigma \)

Medium - pore diameter, \( D \), spacing between pores, \( l \)

Appropriate non-dimensional parameters:

\[ C_a = \frac{\text{Capillary No.}}{\text{Capillary force}} = \frac{\text{Viscous force}}{\text{Capillary force}} \]

\[ M = \text{Mobility Ratio} = \frac{V_1}{V_2} = \frac{k_1n_1 \mu_1}{k_2 n_2 \mu_2} = \frac{\mu_1}{\mu_2} \]

\[ \frac{2 \left( \frac{V_1}{V_2} \right)}{V_2} \]

\( M > 1 \) No viscous fingering

\( M < 1 \) Viscous fingering

\[ B_0 = \text{Bond No} = \frac{\text{Gravity}}{\text{Interfacial}} = \frac{\text{We}}{\text{Fr}} \]

Note: \( C_a \) and \( B_0 \) includes:

\[ \frac{\text{Re}}{\text{We}} \] but some info lost since 2 parameter (groups) from 3 individual.

\( \text{Fr} \)

\( M \) gives information on instability - a new length scale.
IMMISCIBLE DISPLACEMENT

2 is the displacing fluid

Absence of buoyancy effects:
- Viscous fingering when low viscosity fluid displaces a high viscosity fluid.
- Front stabilizes if high viscosity displaces low viscosity.
- Capillary fingering results at low flow rates.

Presence of buoyancy effects:

$$B_0 = \frac{g (\rho_2 - \rho_1) (D/2) l}{\sigma}$$

Gravity

Figure 1. Two-dimensional phase diagram proposed by Lenormand et al. [1988], showing immiscible displacement regimes as a function of their position in $M$, $Ca$ space. (Copyright Cambridge University Press; reprinted with permission.)

Figure 2. Extension of Lenormand et al.'s [1988] phase diagram to three dimensions, showing the hypothesized configuration of the stable flow regime. Lenormand et al.’s [1988] diagram is the $Bo = 0$ plane. Because no physical experiments were performed, values other than zero are deliberately omitted from the vertical axis. (After Bentowitz and Ewing [1998]. Copyright Kluwer Academic Publishers; reprinted with permission.)
Flow - REFRESHER

Permeability (L^2)

Hydraulic conductivity (LT^-1)

\[ \frac{k}{\mu} = \frac{K}{\rho g} \]

\[ Q = -AK \frac{dh}{dx} \quad ; \quad V_a = -k \frac{dh}{dx} \]

\[ V_a = \frac{V_a}{n} \]

RTD (Residence Time Distribution) or "Breakthrough" Curves

Darcy Velocity:

\[ V_d = 1 \text{ m/yr} = \frac{1}{1000} \text{ km/yr} \]

Advection velocity:

\[ V_a = \frac{V_d}{n} = \frac{10^{-3}}{0.1} = 10^{-2} \text{ km/yr} \]

Breakthrough time:

\[ V_a = \frac{L}{t} \quad ; \quad t = L/V_a = \frac{1 \text{ km}}{10^{-2} \text{ km/yr}} = 100 \text{ yr} \]
Fig. 1. Schematic representation of lighter than water LNAPL movement through the unsaturated and into the saturated zone [after Abriola and Pinder, 1985a].

Fig. 2. Schematic representation of heavier than water NAPL movement through the saturated and unsaturated zones.

Fig. 3. Ternary diagram showing the relative permeability of the nonaqueous phase as a function of phase saturations [after Faust, 1985].
Immiscible Transport

Idealized behavior:

If contaminants are immiscible then these processes are important.

True behavior:

1. Fingering results
2. Erratic and unpredictable distribution
3. DNAPL "very" penetrative
   - low viscosity
   - high density
   - low interfacial tension

Questions:

- How far will they migrate?
- What are the controls on penetration?
  - clay-vs-sand aquifers
  - fractured-vs-unfractured
- How do they dissolve/leach/revert?
- How may they be:
  - Remobilized
  - Chemically immobilized
  - Physically isolated.
Flow of IMmiscible Fluids

1. Types of Fluid Flows:

   Miscible displacement:
   - Two fluids completely soluble
   - Concentration
   - No interface: no interfacial tension
   - Hydrodynamic dispersion/diffusion

   Immiscible displacement:
   - Simultaneous flow of two fluids
   - Capillary pressure difference, e.g., Air-water
   - OIl-water (gas)
   - Some mixing at interface is possible
     but likely small.

2. Interfacial Tension and Capillary Pressure:

2.1 Saturation and Fluid Content:

   \[ S_x = \frac{\text{volume of fluid } x \text{ within REV}}{\text{volume of voids within REV}} \]

   \[ S = \frac{V_f}{V_v} \]

   Unsaturated flow; volumetric moisture content, \( \Theta \):

   \[ \Theta = \frac{\text{vol. water in REV}}{\text{bulk vol. of REV}} \neq 1 \]

   \[ \Theta = \frac{V_w}{V_T} \]

   \[ V_T = V_v + V_s \]

   Note that moisture content in soil mechanics is by weight, not volume, i.e.

   \[ m/\ell = \frac{\text{water}}{\text{dry soil}} \]
Figure III.a. Schematic of a small spill in a permeable unsaturated zone with a resulting mound of CHC gas. Concentration decreases with distance from spill. Grainsize shown at right.

Figure III.b. Larger spill than in III.a., but still not large enough to exceed the retention capacity of the unsaturated zone; no liquid CHC reaches the

Figure XVI.a. Beads initially moist; diameter range = 0.85 - 1.23 mm. PER then dripped in from above. The PER accumulated as a sheath around a zone of high water content.

Figure XVI.b. Beads initially saturated with water; diameter range = 0.49 - 0.70 mm. PER then applied from above. When the flow of PER was discontinued, the front portion of the PER stream broke off and halted.
2.2 Interfacial Tension and Wettability

Interfacial tension due to molecular attraction — different in two fluids

Interfacial tension, \( \sigma_{ik} \), between two fluids \( i \) and \( k \).

\( \sigma_{ik} \) is temperature dependent \( \Rightarrow \) Capillary pressure \( = f(T) \).

**Dupré's formula**

\[
W_{ik} = \sigma_i + \sigma_k - \sigma_{ik} \Rightarrow \text{Work to separate into separate i and k with vapor interface}
\]

\( \sigma_i = \text{surface tension of fluid with its own vapor.} \)

\( \sigma_{ik} = \text{interfacial tension.} \)

**Equilibrium**

\[
\sigma_{AB} \quad \sigma_{BG} \quad \sigma_{GQ}\]

Gas or liquid, G

\[
\sigma_{AQ} \quad \text{Liquid, A} \quad \text{Liquid, B}\]

\[
\theta_{AB}\]

\[
\theta_{BG}\]

Equilibrium:

\[
\sigma_{AQ} = \sigma_{AB} \cos \theta_{AB} + \sigma_{BG} \cos \theta_{BG}
\]

Only satisfied if \( \sigma_{AQ} < (\sigma_{AB} + \sigma_{BG}) \)

\( \Rightarrow \) lens of B formed.

If \( \sigma_{AQ} > (\sigma_{AB} + \sigma_{BG}) \)

then B spreads between A and G.

By convention, \( \theta \) measured in denser fluid

Equilibrium:

\[
\sigma_{QL} \cos \theta = \sigma_{SL} - \sigma_{QL}
\]

\[
\cos \theta = (\sigma_{SL} - \sigma_{QL}) / \sigma_{QL}
\]

\( \Rightarrow \) May determine interface angle for known \( \sigma \) (contact angle, \( \theta \)).

- influenced by surface roughness.
If \((\sigma_{sq} - \sigma_{sl})/\sigma_{sl} > 1\) \[\Rightarrow\] No equilibrium

Liquid flows over solid \[\Rightarrow\] wettability

**Wettability** Defined by \(\theta\)

- \(\theta < 90^\circ\): eg. Fluid L wets the solid; wetting fluid
- \(\theta > 90^\circ\): eg. Fluid G, is non-wetting fluid
- \(\theta = 90^\circ\): Zero adhesion tension since equal affinity

Wettability is controlled by:

a) Chemical composition of fluids
   eg. adding dopant

b) Sequence of wetting
   eg. Advancing or receding.

- Hysteresis - wetting or draining

[Diagram]

Water wets solid
Water non-wets solid

Most geologic systems are waxed wet:

- Petroleum/water
- Air/water
- NAPL/water
IMPORTANT ASPECTS OF WETTING

Water-Wet

a) Rendicular rings @ grain contacts
   - no continuous water-phase
   - except mono-molecular coating of H2O

b) ↑ wetting phase saturation →
   - continuous water (wetting) phase
   - equilibrium saturation = when phase is continuous

Implication: wetting phase may be removed.

c) Above critical saturation (equilibrium) funicular saturation
   - Now non-wet phase non-continuous.
   - Can only remove non-wetting phase if large
     pressure gradient to squeeze through
     pore throat.

Funicular saturation critical in remediation!

Remediation @ residual saturations:

- Graves
- Pools

Important since difficult
to remove last
few % of saturation.
Fig. 9.2.1. Interfacial tensions.

Fig. 9.2.2. Contact angle between a water-air interface and a solid. (a) Water wetting the solid. (b) Water nonwetting the solid.

Fig. 9.2.3. Possible fluid saturation states.
2.3 Capillary Pressure

Important: Controls penetration of immiscible fluids

Water-wet system

Note: (a) $p_F = p_g = p_a$ within any phase for static conditions
d as $p_w$ depends on surface curvature

Capillary rise, $h_c$:

$$Y_c T d^2 = 0.5 T d \cos \theta$$

$$h_c = \frac{4 \sigma_d}{d Y_w}$$

Expect: $d \downarrow \Rightarrow h_c \uparrow$
d = pore throat or fracture aperture

Capillary pressure, $P_c$:

$$P_c = P_{nw} - P_w \Rightarrow P_c = \frac{1}{Y_w} (P_{nw} - P_w) = h_c Y_w$$

In the capillary pressure relationship, $h_c = \frac{4 \sigma_d}{d Y_w}$

- the assumption that $\theta \to 0$ is made for clear glass
- Not good for soils/rocks

Most important deduction:

$$P_c = h_c Y_w \propto \frac{\sigma_d}{d}$$

Capillary pressure inversely proportional to pore size, $d$.
Smaller $d$, requires larger $P_c$ to penetrate.

Since many potential pore throat diameters exist,
a capillary model may be replaced by
a grain-grain contact model.
Fig. 9.2.4. Equilibrium at a curved interface between two immiscible fluids.

\[ \Delta p = p_c = p_2 - p_1 = \sigma_{12} \left( \frac{1}{r'} + \frac{1}{r''} \right) = \frac{2 \sigma_{12}}{r^*} = p_c \]

\[ r^* = \text{mean radius} \Rightarrow \frac{2}{r^*} = \left( \frac{1}{r'} + \frac{1}{r''} \right) \quad \text{(Laplace eqn.)} \]

**PROBLEM:**
\[ r^* \] is difficult to determine
- Multiple grain sizes (and pore throat sizes)
- Distribution of pore sizes

\[ \text{Huge} \rightarrow \text{first constriction} \rightarrow \text{free passage} \]

- Pore geometry
- Fluids (\( \sigma_{12} \)) and contact angles (cos\( \theta \))

\[ \therefore \text{Use a statistical average } \Rightarrow \text{Determine } p_c = p(Sw) \] Lab ok

Field ok
**How to determine** $P_c = P_c(S_w)$

For capillary tube of radius, $r$.

$$P_c = \frac{2\sigma \cos \theta}{r}$$

Semi-empirical approach, Leverett (1941). Dimensional analysis gives:

$$J = J(S_w) = \left(\frac{P_c}{\delta}\right)\sqrt{kn} \ ; \ P_c = P_c(S_w)$$

**May also be influenced by** $\cos \theta$ (factor influencing contact angle).

$J$ - Leverett function reduces to a common curve for different materials

$k = \text{permeability (L)}$ \(\sqrt{\frac{k_m}{a}}\) to mean pore diameter

$n = \text{porosity}$

Dependent on formation type

Brooks & Corey (1964)

$$P_c = P_c(S_e) \quad S_e = \frac{(S_w - S_{w_0})}{(1 - S_{w_0})}$$

$S_e = \text{effective saturation}$

$S_{w_0} = \text{irreducible water fluid saturation}$

Gives a straight line relationship in log-log $S_e$ versus $P_c/S_w$ space except close to $S_e = 100\%$. 
**BROOKS-COREY CURVES**

Instead of:

\[ S_e = \frac{(S_w - S_{w_0})}{(1 - S_{w_0})} \]

Use:

\[ S_e = \text{effective saturation} \]

\[ S_{w_0} = \text{irreducible wetting fluid saturation} \]

Curve defined by two parameters:

\[ \lambda = \text{negative slope of curve (pore size distribution)} \]

\[ P_b = \text{intercept of line and } S_e = 100\% \]

also termed "bubbling pressure"

Pressure needed to force a "bubble" of fluid through the pore throat

**REPRESENTED AS:**

\[ S = (1 - S_{w_0})(\frac{P_c}{P_b})^{-\lambda} + S_{w_0} \]

or

\[ S_e = \frac{(S - S_{w_0})}{(1 - S_{w_0})} = \left(\frac{P_c}{P_b}\right)^{-\lambda} \]

i.e.

\[ \log(S_e) = -\lambda \log\left(\frac{P_c}{P_b}\right) \]
Fig. 9.2.6. Leverett function for various formations (Roo and Brown, 1940).

Fig. 9.2.7. Capillary pressure head as a function of effective saturation for porous materials of various pore-size distributions (Brooks and Corey, 1964).
CAPILLARY RISE IN FRACTURES

**Idealized**

\[ w \cdot b \cdot h_c \cdot \gamma_w = 2w_0 \sigma \cos \theta \]

\[ h_c = \frac{2\sigma}{\gamma_w b} \]

**Real** \( P_c = f(S_w) \)

\[ P_c = \frac{2\sigma}{b} \]

\[ P_c \propto \frac{\sigma}{b} \]

\[ \sigma \approx 7 \times 10^{-2} \text{ N/m.} \]
2.4 DRAINAGE AND IMBIBITION

Capillary pressure is hysteretic

- 0 changes with direction of drop.

- For two fluids (liquids)

- For fluid-gas

- Raindrop

Not a unique function of Sw, but depends on path.

Conclude that we cannot determine Pc from Sw alone.

DRAINAGE - Saturated with wetting fluid initially - displace with non-wetting

IMBIBITION - Saturated with non-wetting fluid initially -

place wetting fluid on surface → "spontaneous" imbibition

to equilibrium condition where capillary forces equal those due to gravity.

Unsaturated flows - Water = wetting; air = non-wetting

Sw0 = entrapped air in sample

Sw < Sw0, the non-wetting fluid is non continuous

- does not flow.

Drying and wetting scanning curves are hysteretic.
Fig. 9.2.9. Contact angle ($\theta$) in a capillary tube in a stationary state, in a displacement of a nonwetting liquid by a wetting one ($\theta_1$) and in a displacement of a wetting liquid by a nonwetting one ($\theta_2$).

(a)  
(b)  

Fig. 9.2.9. Hysteresis in contact angle (rain drop effect).

Fig. 9.2.10. Typical capillary pressure—wetting fluid saturation curves illustrating hysteresis.
**Vadose Zone**

\[ P_c = P_{air} - P_{water} \]

**Static Systems**

\[ P_c = P_{nw} - P_w \]

**NAPL in Groundwater**

\[ P_c = P_{NAPL} - P_{water} \]

- **Pc**: Matric Potential (negative pressure, tension)
- **Swc**: Water Saturation
- **Capillary Fringe**: Tension-saturated zone
- Graphs illustrate pressure and saturation relationships.
2.5 Non-Aqueous Fluid Penetration (Static groundwater field)

Question: How deep will a DNAPL penetrate? Will it arrest?
Experience/evidence shows insidious penetration.

\[ P_{co} = \text{entry pressure} \]

Note - wish to penetrate at minimum saturation, not to saturate.
I.e. Penetrate @ low saturation

Consider without porous medium:

Non-wet

Saturating

No penetration unless instability develops

\[ P_{W} = \gamma_{nw} h + \gamma_{w} dh \]

If \( \gamma_{w} < \gamma_{nw} \)
then \( P_{w} > P_{nw} \)

:: Instability develops and pressure difference increases as \( dh \) increases
I.e. Accelerating instability

In porous medium:

\[ P_{co} = \gamma_{nw} h \]

Fluid will penetrate into porous medium. Two possibilities.

1) Flow until all large void space is filled, and stops?
2) Flow, but large void space is sufficiently connected to drain reservoir? \( \checkmark \)
Can flow stop?

Will stop if \( P_{\text{in}} < P_{\text{out}} + P_{\text{w}} \) (1)

\[ h_{\text{in}} \leq h_{\text{w}} + h_{\text{c}} Y_{\text{w}} \] (2)

\[ h_{\text{in}} - h = h_{\text{c}} \frac{Y_{\text{w}}}{Y_{\text{w}}} \] (3)

\[ h_{\text{min}} = \frac{h_{\text{c}}}{(1 - Y_{\text{w}}/Y_{\text{w}})} = \frac{P_{\text{in}}/Y_{\text{w}}}{(1 - Y_{\text{w}}/Y_{\text{w}})} \] (4)

Only limitation is that "phase must be "continuous."

Equation (4) shows: as \( Y_{\text{w}} \) becomes denser, the less high column of non-wetted fluid needed to generate \( P_{\text{co}} \).

**Practical Implications:**

1. Will only stop if (a) Fills "large voids" space and no connected "large void" space remains
   (b) Flow "run-out" even "and h\text{min} is split"
   (c) Fluid system is not "static". What is flow direction.

**Important parameter is the connected pore space!!**

This is difficult to evaluate with conventional test methods.
MAKE SURE YOU UNDERSTAND THE UNDERLYING ASSUMPTIONS

BEFORE YOU APPLY THESE

For invasion:

\[ h_{\text{min}} = \frac{P_{\text{co}}}{k_{\text{aw}}} \left( \frac{1 - Y_{\text{aw}}/Y_{\text{uw}}}{} \right) = \frac{h_c}{(1 - Y_{\text{aw}}/Y_{\text{uw}})} \]

For invasion:

\[ h_{\text{min}} = \frac{P_{\text{co}}}{k_{\text{aw}}} - h_c \]
Table 4.2. Relationships between capillary pressure, gravity, and hydraulic forces useful for estimating conditions of DNAPL movement (from Kueper and McWhorter, 1991; WCGR, 1991; and Mercer and Cuben, 1990).

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Capillary pressure exerted on the surface of a nonwetting NAPL sphere</td>
<td>( P_s = \frac{P_{\text{NAPL}} - P_a}{(\beta \times \cos \phi)} / \tau )</td>
</tr>
<tr>
<td>(b) Capillary pressure exerted on the surface of NAPL in a fracture plane</td>
<td>( P_s = \frac{P_{\text{NAPL}} - P_a}{(\beta \times \cos \phi)} / b )</td>
</tr>
</tbody>
</table>

**Hydrostatic Conditions**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c) Critical height of DNAPL required for downward entry of DNAPL through</td>
<td>( z_s = \frac{(\beta \times \cos \phi)}{\left(\frac{\sigma_{\text{LN}}}{\rho_s - \rho_a}\right)} )</td>
</tr>
<tr>
<td>the capillary fringe (the top of the saturated zone)</td>
<td></td>
</tr>
<tr>
<td>(d) Critical height of DNAPL required for downward entry of DNAPL into</td>
<td>( z_s = \frac{(\beta \times \cos \phi)}{\left(\frac{\sigma_{\text{LN}}}{\rho_s - \rho_a}\right)} )</td>
</tr>
<tr>
<td>the water-saturated base of a lagoon where DNAPL is placed beneath water;</td>
<td></td>
</tr>
<tr>
<td>or, below the water table, for entry of DNAPL into a layer with smaller</td>
<td></td>
</tr>
<tr>
<td>pore openings (assuming top of DNAPL body last missed under incisional</td>
<td></td>
</tr>
<tr>
<td>conditions)</td>
<td></td>
</tr>
<tr>
<td>(e) Critical height of DNAPL required for entry of DNAPL into a water</td>
<td>( z_s = \frac{(\beta \times \cos \phi)}{\left(\frac{\sigma_{\text{LN}}}{\rho_s - \rho_a}\right)} )</td>
</tr>
<tr>
<td>saturated fracture at the base of a lagoon where DNAPL is placed beneath</td>
<td></td>
</tr>
<tr>
<td>water, or, below the water table, for entry of DNAPL into a water-saturated</td>
<td></td>
</tr>
<tr>
<td>fracture having an aperture, ( b_a ) smaller than the host medium pore</td>
<td></td>
</tr>
<tr>
<td>radius; or, above the water table, or entry of DNAPL into a water-saturated</td>
<td></td>
</tr>
<tr>
<td>fracture segment having an aperture smaller than that of the overlying</td>
<td></td>
</tr>
<tr>
<td>host fracture segment (assuming top of DNAPL body last missed under</td>
<td></td>
</tr>
<tr>
<td>incisional conditions)</td>
<td></td>
</tr>
<tr>
<td>(f) Critical height of DNAPL required below the water table, for entry of</td>
<td>( z_s = \frac{(P_{\text{NAPL}} - P_{\text{dimm}})}{\left(\frac{\sigma_{\text{LN}}}{\rho_s - \rho_a}\right)} )</td>
</tr>
<tr>
<td>DNAPL into a layer with smaller pore openings where the top of the DNAPL</td>
<td></td>
</tr>
<tr>
<td>body is under incisional conditions</td>
<td></td>
</tr>
</tbody>
</table>

**Hydraulic Conditions**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g) The stable DNAPL pool length, ( L_p ), that can exist below the</td>
<td>( L_p = \frac{(\alpha \times \cos \phi)}{\left(\frac{\sigma_{\text{LN}}}{\rho_s - \rho_a}\sin \theta\right)} )</td>
</tr>
<tr>
<td>water table following initial DNAPL migration where ( \theta ) is the</td>
<td></td>
</tr>
<tr>
<td>dip angle of the fracture, ( b ) is the maximum fracture aperture at the</td>
<td></td>
</tr>
<tr>
<td>leading edge of the DNAPL pool, and ( L_p ) is measured parallel to</td>
<td></td>
</tr>
<tr>
<td>the fracture slope</td>
<td></td>
</tr>
<tr>
<td>(h) The stable DNAPL pool length, ( L_p ), within a fracture that can</td>
<td>( L_p = \frac{(\alpha \times \cos \phi)}{\left(\frac{\sigma_{\text{LN}}}{\rho_s - \rho_a}\sin \theta\right)} )</td>
</tr>
<tr>
<td>exist below the water table following initial DNAPL migration where ( \theta</td>
<td></td>
</tr>
<tr>
<td>is the dip angle of the fracture, ( b ) is the maximum fracture aperture</td>
<td></td>
</tr>
<tr>
<td>at the leading edge of the DNAPL pool, and ( L_p ) is measured parallel</td>
<td></td>
</tr>
<tr>
<td>to the fracture slope</td>
<td></td>
</tr>
</tbody>
</table>

**Hydraulic Conditions**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) Neglecting capillary pressure effects, the critical upward hydraulic</td>
<td>( \frac{\beta}{\rho_s - \rho_a} = \frac{\alpha}{\gamma_s} )</td>
</tr>
<tr>
<td>gradient, ( \alpha ), required across a DNAPL body of height ( z_s ),</td>
<td></td>
</tr>
<tr>
<td>to prevent downward DNAPL migration in a uniform porous medium</td>
<td></td>
</tr>
<tr>
<td>(j) Neglecting capillary pressure effects, the minimum hydraulic head</td>
<td>( \Delta h = \frac{(\beta - \rho_a) g z_s}{\rho_s} )</td>
</tr>
<tr>
<td>difference between the bottom and top of a DNAPL body of height ( z_s ),</td>
<td></td>
</tr>
<tr>
<td>to prevent downward DNAPL migration in a uniform porous medium</td>
<td></td>
</tr>
<tr>
<td>(k) Neglecting capillary pressure effects, the critical hydraulic gradient</td>
<td>( \frac{\beta}{\rho_s - \rho_a} \cos \theta )</td>
</tr>
<tr>
<td>( \beta ), required to prevent the downward movement of DNAPL along the</td>
<td></td>
</tr>
<tr>
<td>top of a dipping (angle = ( \theta )) capillary barrier (i.e., in sloping</td>
<td></td>
</tr>
<tr>
<td>fractures, bounding planes, or within a sloping course layer above a fine</td>
<td></td>
</tr>
<tr>
<td>gravel layer with ( \beta ) measured parallel to the slope</td>
<td></td>
</tr>
<tr>
<td>(l) Critical horizontal hydraulic gradient, ( \beta ), which must exist</td>
<td>( \beta = \frac{(\beta - \rho_a) g z_s}{\rho_s} )</td>
</tr>
<tr>
<td>across a DNAPL pool of length ( L ) beneath the water table to overcome</td>
<td></td>
</tr>
<tr>
<td>capillary resistance and mobilize DNAPL in the pool (to calculate ( \beta )</td>
<td></td>
</tr>
<tr>
<td>for a pool of DNAPL in a horizontal fracture, replace with the fracture</td>
<td></td>
</tr>
<tr>
<td>aperture, ( b )</td>
<td></td>
</tr>
<tr>
<td>(m) The critical upward hydraulic gradient, ( \alpha ), required to</td>
<td>( \Delta h = \frac{(\beta - \rho_a) g z_s}{\rho_s} + \left[\frac{\beta g z_s (\beta - \rho_a)}{\rho_s}\right] )</td>
</tr>
<tr>
<td>arrest the downward migration of DNAPL through an aquitard of thickness, ( dx ),</td>
<td></td>
</tr>
<tr>
<td>where ( \beta ) is the capillary pressure of DNAPL pooled at the top of the</td>
<td></td>
</tr>
<tr>
<td>aquitard minus the threshold entry (displacement) pressure of the aquitard</td>
<td></td>
</tr>
</tbody>
</table>
2.6 EVALUATION OF CAPILLARY PRESSURE CURVES

**Theoretical evaluation** - Difficult except for uniform rods/spheres

Use $P_c = \frac{2\sigma}{r} \cos \theta$

eg. Collins (1961) for packed rods

$R =$ radius

Limited Sw range since result valid only until adjacent fluid interface contact.

**Laboratory Methods**

$P_c = P_c(Sw)$

a) Displacement methods: Establish successive states of hydrostatic equilibrium

b) Dynamic methods: Establish successive states of hydraulic steady flow

Both with a wetting and non-wetting fluid.

**Displacement methods**

Porous diaphragm method:

1. Wetted core inside non-wetting fluid.

   Underlain by diaphragm dii-permeable but not to the non-wetting fluid.

2. Increase pressure of non-wetting (displacing) fluid and wait for equilibrium (saturating)

   Trace curve with: 1, 2, 3, etc.

3. Determine saturation from displaced fluid or X-ray/CT scan.

   Limitations: 1. Length of time for equilibrium. 10-20d per point? 2. Is all pore space being accessed? Does it matter?

**Advantages:** Use `and' fluids (of interest).
Fig. 9.2.13. Capillary pressure in a cubic packing of circular rods with liquid-air interfaces (after Collins, 1961).
Fig. 9.2.14. Schematic diagram of a porous diaphragm device for capillary pressure determination (Welge and Bruce, 1947).

Fig. 9.4.4. The use of mercury tensiometer manometers. (a) Measurement of capillary pressure in the field, (b) the Haiies apparatus for determining $p_s = p_e(S_o)$. 

\[ p_s = \gamma_m h_2 - \gamma_w h_1 + z \]
3. **Simultaneous flow of two immiscible fluids**

- Capillarity and capillary pressures govern the "equilibrium" penetration of fluids — static behavior
- Once penetrated, the individual phases may transfer and be transported
  - Pure phase (free product)
  - Dissolved form (later).

![Diagram of actual vs. idealized flow](image)

- Note that flow within phase is not subject to capillarity (capillarity acts at fringes only)
- Each fluid establishes its own "tortuous" path to stable channels

![Diagram of NW flow](image)

- No flow of NW
  - (Discontinuous NW)
- Both fluids flow
- No flow of W
  - (Discontinuous Wetting)

*Increasing saturation of Non-Wetting*
3.1 Motion Equations

Apply Darcy's Law

\[ q_1 = \left( \frac{k_1}{\mu_1} \right) \frac{\Delta P_1}{L} \]

\[ q_2 = \left( \frac{k_2}{\mu_2} \right) \frac{\Delta P_2}{L} \]

Establish steady flow at volumetric flow rates, \( q_1 \) and \( q_2 \).

\( q_1 = \left( \frac{k_{1e}}{\mu_1} \right) \frac{\Delta P_1}{L} \) ; \( q_2 = \frac{Q_{2e}}{A_2} \)

\( \Delta P_e \) = effective pressure drop of \( e \)th fluid

\( k_e \) = "effective" permeability of medium to fluid \( e \).

\( k_e \) depends on:

a) Pore size and distribution and fractures.

b) Saturation, \( S_1 \) and \( S_2 \).

"Relative" permeabilities

\[ k_{r1} = \frac{k_1}{k} (S_1) \] \( \uparrow \) \[ k_{r2} = \frac{k_2}{k} (S_2) \]

Relative to single phase permeability of a fully saturating fluid.

Determine from "equilibrium" laboratory flow tests under different saturations.

Effective permeability \( k_e \)

\( k_e = k_{r1} k \)

Time permeability \( (L^2) \)

Relative permeability
Fluid Motion Equation (Darcy's Law)

\[ q_{i1} = -k \frac{\rho_1 \frac{\partial p_1}{\partial x_j}}{\mu_1} - k \frac{\mu_1}{\mu_1} \left( \frac{\partial p_1}{\partial x_j} + \rho_1 g \frac{\partial z}{\partial x_j} \right) \]

\[ q_{i2} = -k \frac{\rho_2 \frac{\partial p_2}{\partial x_j}}{\mu_2} - k \frac{\mu_2}{\mu_2} \left( \frac{\partial p_2}{\partial x_j} + \rho_2 g \frac{\partial z}{\partial x_j} \right) \]

May also use pressure head, \( h \), for constant \( p \) and \( \rho_2 \), but must be defined separately for each fluid, \( h_k \)

\[ h_k = \frac{p_k}{\rho_k g} + z \quad j = 1, 2 \]

\[ q_{i1} = -k \frac{\mu_1}{\mu_1} \rho_1 g \frac{\partial h_1}{\partial x_j} \]

\[ q_{i2} = -k \frac{\mu_2}{\mu_2} \rho_2 g \frac{\partial h_2}{\partial x_j} \]

Similarity between:

\[ q = -k \frac{\rho g}{\mu} \frac{dh}{dx_j} \]

\[ = -k \frac{dh}{dx_j} \]

Hydraulic Conductivity (L/T)
Fig. 9.2.10. Hassler's apparatus for relative permeability determination (after Osoba et al., 1981).

Fig. 9.3.1. Typical relative permeability curves (e.g., wetting fluid = water, nonwetting fluid = oil).
3.2 RELATING PERMEABILITY

- $k_{r_{nw}} + k_{r_w} \neq 1 \Rightarrow k_{r_{nw}} + k_{r_w} < 1$
  
  Most effective transmission is at 100% saturation (if accessible), interference.

- Usually $k_{r_{nw}}$ closer to 1 than $k_{r_w}$

- Steep decline of $k_{r_{nw}}$ with increasing $S_{nw}$ indicates larger pores occupied first by nonwetting phase.

  Nonwetting phase occupies larger pores preferentially due to capillary pressure arguments.

- $k$ to wetting fluid is always larger for open-pored unconsolidated material.

- $k$ to non-wetting fluid is always smaller for open-pored unconsolidated material.

Hysteresis:

1. Wetting fluid surrounds grains and non-wetting fluid may move non fluid even if no pressure gradient in non fluid.

2. Since change in saturation requires change in wetted grain surface - wettability is hysteretic. i.e. permeabilities are hysteretic.
PERMEABILITY/CONDUCTIVITY OF FRACTURES

Flow in fractures:

\[ \bar{V} = \frac{b}{V} \rightarrow Q \]

\[ \bar{V} = \text{average velocity} \]

\[ V = \text{kinematic viscosity of fluid} \]

\[ v = \frac{\Delta \rho}{\rho} \]

For single fracture:

Equivalent flow rate per unit width:

\[ Q = b \frac{g b^2}{12v} \frac{dh}{dx} \]

Multiple fractures arranged in parallel:

Total of \( N \) fractures per unit height:

\[ N = \frac{1}{s} \]

\[ Q = \frac{9b^3}{6\nu s} \frac{dh}{dx} \]

Equivalent conductivity for multiple sets:

\[ K = \frac{g b^3}{6\nu s} \]

Enables \( b \) to be evaluated if \( K \) known (measured).

Relative permeability of fractures:

- Similar behaviour to porous medium
- Distribution of apertures
- Large apertures saturate first

Capillary pressure or head:

\[ h_c = \frac{2\sigma}{b \gamma_w} \]
Fracture permeabilities and capillary pressures

\[ k = \frac{b^2}{12} \quad \text{singly fractured permeability} \quad (1) \]

\[ k_b = \frac{b^3}{12s} \quad \text{bulk permeability} \]

Permeabilities and capillary pressures related.

\[ h_c = \frac{2\sigma}{b\gamma_w} \quad \text{or} \quad h_c \gamma_w = \frac{2\sigma}{b} = P_c_b \quad (2) \]

From (1) and (2)

\[ P_c_b = \frac{2\sigma}{\sqrt{12k}} \]

General relation for fractures and porous media

\[ P_c \propto \frac{1}{\sqrt{1/k}} \]

eg. Lambe's J' function -

\[ J = \frac{P_c}{\sigma \sqrt{1/k}} \]

\[ \therefore P_c = J\sigma \sqrt{1/k} \]
3.3 Mass Conservation in Multiphase Flow

Continuity equation: \[ \frac{\partial}{\partial t} \left( n \, S \, \rho_\alpha \right) + \frac{\partial}{\partial x_i} \left( \rho_\alpha \, q_{\alpha i} \right) = 0 \]

For an incompressible fluid and medium: \[ \frac{\partial}{\partial t} \left( n \, n_\alpha \right) = 0 \]

Substitute \( q_{\alpha i} \) from relative permeability relation:

Results in 4 equations:

\[ n \, \frac{\partial S_1}{\partial t} - \frac{\partial}{\partial x_i} \left[ k \, k_1 \left( \frac{\partial p_1}{\partial x_j} + \rho_1 \, g \, \frac{\partial z}{\partial x_j} \right) \right] = 0 \]

\[ n \, \frac{\partial S_2}{\partial t} - \frac{\partial}{\partial x_i} \left[ k \, k_2 \left( \frac{\partial p_2}{\partial x_j} + \rho_2 \, g \, \frac{\partial z}{\partial x_j} \right) \right] = 0 \]

\[ S_1 + S_2 = 1 \]

\[ p_2 - p_1 = p_c(S_1) \]

Solve for 4 unknowns: \( S_1, S_2, p_1, p_2 \)

with \[ h_1 = z + \frac{p_1}{\rho_1} \] \[ h_2 = z + \frac{p_2}{\rho_2} \]

Solve using numerical techniques.
3.5 Buckley-Leverett (1942) Equations

Assume: Neglect gravity, capillarity, liquid compressibility.

Homogeneous reservoir of thickness $b$, inclined at $\alpha^\circ$.

\[ q_w = -\left(\frac{k_w}{\mu_w}\right)\left(\frac{\partial p_w}{\partial x} + \rho_w g \sin \alpha \right) \]

\[ q_{nw} = -\left(\frac{k_{nw}}{\mu_{nw}}\right)\left(\frac{\partial p_{nw}}{\partial x} + \rho_{nw} g \sin \alpha \right) \]

\[ \frac{\partial S_w}{\partial t} + \frac{\partial q_w}{\partial x} = 0 \] \hspace{1cm} (1)

\[ \frac{\partial S_{nw}}{\partial t} + \frac{\partial q_{nw}}{\partial x} = 0 \] \hspace{1cm} (2)

\[ S_w + S_{nw} = 1 \] \hspace{1cm} (3)

\[ P_c = P_{nw} - P_w \] \hspace{1cm} (4)

Initial and boundary conditions

\[ t \leq 0 \hspace{1cm} 0 \leq x \leq L \hspace{1cm} S_{nw} = 1 - S_{w_0} \]

\[ t > 0 \hspace{1cm} x = 0 \hspace{1cm} q_w = \text{constant} \hspace{1cm} q_{nw} = 0 \]

Wetting fluid injected to displace non-wetting fluid (oil) at irreducible wetting saturation $S_{nw} = 1 - S_{w_0}$

Injected at constant rate at $x = 0$. 
Unconsolidated - open pores
- high K to mobile fluid
- low K to non-mobile

Fig. 9.1.2. Typical relative permeability to gas and water. (a) Unconsolidated sand. (b) Consolidated sand (Biot, 1941).

Inject water
- \( x = 0 \)
- \( t = 0 \)

Fig. 9.3.3. Effect of hysteresis on relative permeability.

\[ \frac{dx}{\sin \theta} = \frac{dz}{\lambda} \Rightarrow \frac{dz}{dx} = \sin \theta \]

\[ r_w = \frac{1}{1 + \frac{r_{nw} r_{uw}}{r_{nw}}} \]

Fig. 9.3.4. The Buckley-Leverett solution for a linear immiscible displacement.
B-L Equations (cont'd)

Enables evaluation of front movement, see figure 9.3.4.(e)

\[ \frac{r_w}{1 + \frac{r_k}{r_m}} = \frac{1}{1 + \frac{1}{(r_k/r_m)_{\text{ratio}}}} \]

Note since capillarity neglected \( t_4 \), two saturations exist.
- only applicable to high-flow rates
  - where capillary effects are masked

\[ M = \frac{r_k}{r_m} = \frac{V_w}{V_m} \]

General Comments

1. Techniques developed for petroleum industry
   Difficult interpretation for saturations less than irreducible saturation. This is perhaps most important region for groundwater contamination

   Need to incorporate:
   - Dissolution effects.

2. For pump-and-treat need to understand two effects:
   a) Role of \( k_r \) in controlling water flow rates through NAPL plug. Plug does not move since capillary water saturation
   b) Role of dissolution as \( S_m \) increases

[Diagram showing NAPL plug and dissolution plume]
Fig. 2 - Results with Mass Transfer Coefficient = 1x10^-4 \( \nu_x \) (a) Concentration at early times (b) Concentration at late times (c) DNAPL Saturation (d) Mass distribution (e) Relative permeability.

COLUMN EXPERIMENT

DNAPL Residual Zone

SOLVE FOR:
- HEAD
- CONC.
- DNAPL
- SAT.
- REL. K

Flow Direction

\( \frac{dh}{dx} = 0.1 \)
Fig. 3 - Results with Mass Transfer Coefficient $= 1 \times 10^{-5}$ $\text{h}^{-1}$

Fig. 4 - Results with Mass Transfer Coefficient $= 2 \times 10^{-5}$ $\text{h}^{-1}$

Fig. 5 - Concentration observed at the end of the residual saturation zone for different Mass Transfer Coefficients and lengths of the residual saturation zones.
3.6 Non-Aqueous Fluid Poretration (Moving Groundwater)

Two situations: 1) NAPL saturates medium → No capillary forces  
               2) NAPL and water

3.6.1 NAPL Saturated

What pressure gradient to move fluid?  
What direction of movement?

\[ \text{Body Force} \rightarrow \text{Seepage Force} \rightarrow \text{Resultant} \]

\[ \text{d}V/\rho g \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \text{d}V \begin{bmatrix} -\frac{\partial \psi}{\partial x} \\ -\frac{\partial \psi}{\partial y} \\ -\frac{\partial \psi}{\partial z} \end{bmatrix} \text{P} = \text{F} \]

\[ \left[ \rho g \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} - p \begin{bmatrix} \frac{\partial \psi}{\partial y} \\ \frac{\partial \psi}{\partial z} \end{bmatrix} \right] \text{d}V = \text{F} \]

\[ \text{equilibrium} \quad \rho \frac{\partial ^2 \psi}{\partial t^2} = -\rho g \]

\( F \) is the resultant force vector, torque per unit volume.
Define the direction of force and \( z \) direction of movement.

\( F_z = 0 \)  \( \frac{\partial p}{\partial z} = -\rho g \) static fluid
\( F_z = \text{the} \)  \( \frac{\partial p}{\partial z} < -\rho g \) ↑ flow
\( F_z = -\text{the} \)  \( \frac{\partial p}{\partial z} > -\rho g \) ↓ flow

Also define lateral direction of flow.
Will the gradient move the NAPL?

Require to overcome capillary pressure, $P_c$.

$P_c$ acts uniformly in all directions and must be overcome.

Need to know size of NAPL pocket.

$T$ acts over length, $L$.

$L$ is typically unknown.
3.7 Behavior of LNAPLS

2.7.1 Migration of LNAPLS

Surface spill:

1. Penetrate vadose zone under gravity with influence of capillary forces.
   - Vadose zone water wet.
   - NAPL is non-wetting.

   Capillary Fringe: Holds water as continuous phase with residual air saturation.
   - \( h_c \uparrow \text{ with } \downarrow \text{ grain size } \rightarrow \downarrow \text{ pore size} \)
   - \( \text{i.e. } h_c = \frac{4\gamma}{\delta} \)

Example usual capillary rise:

\begin{align*}
\text{Fine gravel} & \quad 2-5 \quad 2.5 \\
\text{Medium sand} & \quad 0.2-0.5 \quad 24.6 \\
\text{Fine silt} & \quad 0.02-0.05 \quad 200+
\end{align*}

- LNAPL will penetrate down to top of capillary zone (if enough volume)

TABLE 5.1 Visual capillary rise in unconsolidated materials (porosity of all samples is about 41%).

<table>
<thead>
<tr>
<th>Material</th>
<th>Grain Size (mm)</th>
<th>Capillary Rise (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fine gravel</td>
<td>2–5</td>
<td>2.5</td>
</tr>
<tr>
<td>Very coarse sand</td>
<td>1–2</td>
<td>6.5</td>
</tr>
<tr>
<td>Coarse sand</td>
<td>0.5–1</td>
<td>13.5</td>
</tr>
<tr>
<td>Medium sand</td>
<td>0.2–0.4</td>
<td>24.4</td>
</tr>
<tr>
<td>Fine sand</td>
<td>0.1–0.2</td>
<td>42.8</td>
</tr>
<tr>
<td>Silt</td>
<td>0.05–0.1</td>
<td>105.5</td>
</tr>
<tr>
<td>Fine silt</td>
<td>0.02–0.05</td>
<td>200+</td>
</tr>
</tbody>
</table>

Source: Lohman (1972).

FIGURE 5.14 Changes in the vertical distribution of oil with time after a slug of oil is added to the top of a column of sand. Oil content and water content are expressed as a fraction of the total volume of the porous media. Source: D. K. Eckberg and D. K. Simada, Water Resources Research 20, no. 12 (1984): 1891–97. Copyright by the American Geophysical Union.
DEVELOPMENT OF AN OIL TABLE (Abdul, 1988)

Add oil incrementally to:

- **a)** develop oil fringe
- **b)** develop oil table
- **c)** remove capillary fringe and create oil table.

Mobile oil product may migrate "free" product

**3 Components**

- **Residual column**
  - What is saturation? — Since free phase moved through zone, residual saturation must be greater than irreducible.
  - Re-mobilized as rain fall from surface &
  - Partitioning → 1. Vapor phase → volatilization.
  - 2. Dissolved in aqueous phase.

- "Free" product — May mobilize with rising/falling water table. May flow down steep water table.

- **Dissolution into groundwater**
  - Gasolines → Benzene, Toluene, Ethyl benzene & Xylenes (RTX) soluble fractions.
  - Degree of partitioning depends on volatility of fraction

Which fraction will partition in → air? 
=> water? Henry’s Law

What is partitioning rate?

→ Transport processes (aqueous/gaseous).
FIGURE 5.15 Hydrostatic pressure head/tension distribution in a sand column to which oil is being added to the top of the column. (a) Before the addition of the oil, (b) after addition of the oil showing the development of an oil fringe, (c) after addition of sufficient oil for an oil table to form, and (d) after sufficient mobile oil has accumulated to eliminate the water capillary fringe. Source: A. S. Abdul, Ground Water Monitoring Review 8, no. 4 (1988): 73–81. Copyright © 1988 Water Well Journal Publishing Co. Used with permission.

FIGURE 5.16 Subsurface distribution of an LNAPL spill.
Henry's Law

Linear relationship between vapor pressure of a solute and its aqueous solution and the concentration of the solution. Proportionality coefficient = Henry's Law constant.

Also applicable to partitioning in air:

Low water-air partition coefficients (alkanes) → favor vapor phase
High water-air " " (benzene) → favor aqueous phase

Different time of release for different materials:

-潜水
-汽油蒸气
-土壤溶剂
-溶解
FIGURE 5.17 Mass of residual hydrocarbon in vadose zone partitioning into capillary water as a function of time, with (a) total hydrocarbons, (b) aromatic constituents, and (c) nonaromatic constituents. Source: A. L. Boehm, Water Resources Research 23, no. 10 (1987): 1926–38. Copyright by the American Geophysical Union.

<table>
<thead>
<tr>
<th>Compound</th>
<th>Formula</th>
<th>Molecular Weight</th>
<th>Water-Air Partition Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aromatics</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Benzene</td>
<td>C₆H₆</td>
<td>78</td>
<td>5.66</td>
</tr>
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<td>Toluene</td>
<td>C₇H₈</td>
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<td>3.85</td>
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<td>o-Xylene</td>
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<td>4.66</td>
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<tr>
<td>Ethylbenzene</td>
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<td>3.80</td>
</tr>
<tr>
<td>Nonaromatics</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cyclohexane</td>
<td>C₆H₁₂</td>
<td>84</td>
<td>0.15</td>
</tr>
<tr>
<td>1-Hexene</td>
<td>C₇H₁₂</td>
<td>84</td>
<td>0.067</td>
</tr>
<tr>
<td>n-Hexane</td>
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<td>0.015</td>
</tr>
<tr>
<td>n-Octane</td>
<td>C₉H₂₀</td>
<td>114</td>
<td>0.0079</td>
</tr>
</tbody>
</table>

3.7.2 Thickness of Floating Product

What is true depth/thickness of NAPL?
Borehole will show artificial LNAPL depth
Artificial depth sense.

WELL THICKNESS (of product)
Water capillary fringe

Water

\[ P_0 = P_w \]
\[ P_0 = \rho_0 T g \]
\[ P_w = \Delta \rho w W g \]

Equation \[ \rho_0 T = \rho w W \]
\[ \Rightarrow \]
\[ W = \frac{T}{\Delta \rho w} \]
Measure \( T \), calculate \( W \).

Volume of product \( \Rightarrow \) Thickness \((T-W)\) (ignoring cap fringe height)
Volume of porespace \[ A(T-W) n \]
Volume of product \[ A(T-W) n (1-S_{w0}) \]

Not all is recoverable!!

Approx recoverable volume of product \[ A(T-W) n (1-S_{w0} - S_{w0}) \]
Farr, Houghtalen, and McWhorter (1990) and Lehnard and Parker (1990) developed two methods to estimate the volume of recoverable LNAPL in an aquifer based on the thickness of the LNAPL floating in a monitoring well. These methods are based on the capillary soil properties. One of the two methods is based on the determination of soil properties as reported by Brooks and Corey (1966). We will look at this method in some detail using the derivation of Farr, Houghtalen, and McWhorter.

As shown in Figure 5.19 is the difference between the depth to the water-oil interface in the well, $D^w_w$, and the depth to the oil-air interface, $D^o_w$. The values of the depth to the oil table in the aquifer, $D^o_a$, and the depth to the top of the capillary fringe, $D^w_a$, can be computed.

\[
D^w_a - D^o_a = \frac{P^w_w}{\rho_w g} - \frac{P^o_a}{\rho_a g}
\]

(5.30)

where

- $P^w_w$ = the Brooks-Corey air-organic displacement pressure
- $P^o_a$ = the Brooks-Corey organic-water displacement pressure
- $g$ = the acceleration of gravity

Recall

\[
S_e = \frac{e}{e_{s H_{w, C}}}
\]

\[
S_e = \left( \frac{S_m - S_m^0}{1 - S_m^0} \right)
\]
FIGURE 3.19 Comparison of distribution of mobile oil in an aquifer with the thickness of floating oil in a monitoring well for the case where a water capillary fringe exists below the zone of mobile oil.

TERMINOLOGY: 
- $D_w$: Measured in the well
- Organic-water interface
Equation 5.30 may be rewritten as

$$D_e^{ow} = (D_e^{oc} + T) - \frac{p_e^{ow}}{(\rho_w - \rho_a)g}$$  \hspace{1cm} (5.31)

If any of the organic liquid exists at a positive pore pressure, then $D_e^{ow}$ will be greater than $D_e^{oc}$ and from Equation 5.31,

$$T \geq \frac{p_e^{ow}}{(\rho_w - \rho_a)g}$$  \hspace{1cm} (5.32)

If the organic liquid is all under tension in the capillary zone, then there will be no mobile organic layer and no organic liquid will collect in the monitoring well. Under these conditions, Equations 5.29, 5.30, 5.31, and 5.32 are not applicable. However, as soon as free organic liquid appears in the aquifer, it will collect to a depth of at least $p_e^{ow} / (\rho_w - \rho_a)g$.

The total volume of nonresidual organic liquid in the vadose zone is given by

$$V_e = n \left\{ \int_{D_e^{ow}}^{D_e^{oc}} (1 - S_w) \, dz - \int_{D_e^{ow}}^{D_e^{oc}} \left[ (1 - (S_o + S_l)) \right] \, dz \right\}$$  \hspace{1cm} (5.33)

where

- $V_e$ = the volume of organic liquid per unit area
- $n$ = the porosity
- $S_w$ = the water saturation ratio
- $S_o$ = the organic liquid saturation ratio
- $z$ = the vertical coordinate measured positively downward
- $D_e^{ow}$ = a value determined from Equation 5.30
- $D_e^{oc}$ = a value determined from Equation 5.29
- $D_e^{ow}$ = the top of the zone where nonresidual oil occurs

Based on work by Lenhard and Parker (1987, 1988), the fluid-content relations are

$$S_o - S_w = (1 - S_w) \left( \frac{p_e^{ow}}{p_e^{oc}} \right) \frac{1}{\lambda} + S_{wi} \quad p_e^{oc} > p_e^{ow} \hspace{1cm} (5.34a)$$

$$S_o + S_w = 1, \quad p_e^{ow} < p_e^{oc} \hspace{1cm} (5.34b)$$

$$S_o = (1 - S_w) \left( \frac{p_e^{ow}}{p_e^{oc}} \right) \frac{1}{\lambda} + S_{wi}, \quad p_e^{ow} > p_e^{oc} \hspace{1cm} (5.35a)$$

$$S_o = 1, \quad p_e^{ow} < p_e^{oc} \hspace{1cm} (5.35b)$$

where

- $S_{wi}$ = the irreducible water saturation
- $\lambda$ = the Brooks-Corey pore-size distribution index

In addition,

$$p_e^{ow} = \rho_a g (D_e^{ow} - (p_e^{ow} / \rho_a g) - z) + p_e^{ow}$$  \hspace{1cm} (5.36)

$$p_e^{ow} = g (\rho_w - \rho_o) \left[ \frac{p_e^{ow}}{(\rho_w - \rho_o)g} - z \right] + p_e^{ow}$$  \hspace{1cm} (5.37)
Integration of Equation 5.33 for \( D^\text{new}_w > 0 \), using Equations 5.34, 5.35, 5.36, and 5.37, yields the following. For \( \lambda \) not equal to 1,

\[
V_\text{v} = \frac{(1 - S_{\text{w}})D}{1 - \lambda} \left[ \lambda + (1 - \lambda) \left( \frac{T}{D} \right) - \left( \frac{T}{D} \right)^{1 - \lambda} \right]
\]

(5.38a)

For \( \lambda \) equal to 1,

\[
V_\text{v} = n(1 - S_{\text{w}})[1 - D(1 + \ln T)]
\]

(5.38b)

where

\[
D = \frac{p_i^w}{(\rho_i - \rho_w) g} - \frac{p_i^w}{\rho_w g}
\]

\[
T = D^\text{new}_w - D_w^t \geq \frac{p_i^w}{(\rho_i - \rho_w) g}
\]

If organic liquid above the residual saturation exists all the way to the land surface, then \( D_w^t \) does not exist. Under this condition integration of Equation 5.33 yields the following. For \( \lambda \) not equal to 1,

\[
V_\text{v} = n(1 - S_{\text{w}}) \left\{ (T - D) - \frac{p_i^w}{\rho_i g(1 - \lambda)} \left[ 1 - \left( \frac{\rho_i g D^\text{new}_w}{p_i^w} \right)^{1 - \lambda} \right] \right\}
\]

\[
+ \frac{p_i^w}{(\rho_i - \rho_w) g(1 - \lambda)} \left[ 1 - \left( \frac{\rho_i g D^\text{new}_w}{p_i^w} \right)^{1 - \lambda} \right] \right\}
\]

(5.39a)

For \( \lambda \) equal to 1,

\[
V_\text{v} = n(1 - S_{\text{w}}) \left\{ (T - D) - \frac{p_i^w}{(\rho_i - \rho_w) g} \ln D_w + \frac{p_i^w}{\rho_i g} \ln D_w^t \right\}
\]

(5.39b)

Approximate volume (simplified) = \( V \approx n(1 - S_{\text{w}} - S_{\text{nu,b}})(T - W) \)

Reasons not to be able to recover free product:

1. Lens of low conductivity
3.1.3. Rise and Fall of Water Table

Fall of water table — "Fine" product drops
Rise of water table — Residual LNAPL is trapped below water table and available → dissolution

Rate of movement controlled by: • Darcy's law, $kr/k_n$
  • Density (driving flow)
  • Capillarity

Residual volume controlled by soil retention capacity
  Ground $5 \ell/m^2$
  Silty sand $40 \ell/m^2$

\[
\\text{Say } n = 30\% \quad 1\text{m}^3 = 1000\text{L}
\]

\[
\text{porosity } (\infty) 30\% \rightarrow 20\%
\]

\[
\text{Saturation } S_{w0} = \frac{5}{300} \rightarrow \frac{40}{300}
\]

\[
S_{w0} = 1.6\% \rightarrow 13\%
\]

• Free product may be drained by open wells and trenches
• Bound product must be removed by → volatilization/heating → dissolution

Note: cannot apply high fluid suction pressures due to short circuiting.
FIGURE 5.22 Effect of a falling and then rising water table on the distribution of mobile and residual phases of an LNAPL.
3.8 Behavior of DNAPLs

3.8.1 Vadose zone migration

- Similar characteristics to LNAPL but higher density → more penetrative (water table)

- Preferentially chooses large pores and continuous pore throat channels

- Displaces air and reaches capillary zone

- Displaces water at capillary zone

3.8.2 Vertical movement in saturated zone

Static fluid: Holzer’s formula - defines required critical height for penetration, \( h_o \)

\[
h_o = \frac{2\sigma \cos \theta \left( \frac{1}{r_c} - \frac{1}{r_p} \right)}{g(\rho_w - \rho_o)}
\]

\( r_p \) = pore radius

\( r_c \) = throat radius

\( h_c = \frac{P_b}{Y_0} \)

Rhombohedral packing, \( r_p = \frac{1}{3} d \)

\( r_c = \frac{1}{2} d \)

Same as \( h_c = \frac{h_c}{(1 - P_w/\rho_w)} \)

\( h_c \) in cluded \( \theta, \sigma, \gamma_w, \gamma_m \)

Aquifer

Fractures

Aquitard - will it penetrate?

Capillary tube: \( h_c = \frac{4\sigma \cos \theta}{\gamma_w} \)

Fracture: \( h_c = \frac{2\sigma \cos \theta}{\gamma_w} \)
Monitor wells:

- Place at aquifer base to collect flow from mobile DNAPL.
  - Will separate in well into individual compounds.
  - Water at saturation (potentially).

- DNAPL level is intermediate within zone of mobile water and DNAPL.
- Deep wells record pulse DNAPL depth.

Vertical distribution of DNAPL

Accumulation depends on grain size (actually pore size) distribution → spine controls capillary pressure distribution.

Small pores/fractures: Low K → thin pure DNAPL layer
  - Thick DNAPL + water

Large pores/fractures: High K → thick pure DNAPL layer
  - Thin DNAPL + water
FIGURE 5.23 Zones of a DNAPL and the relationship of mobile DNAPL and nonmobile DNAPL to the DNAPL saturation; relationship of mobile DNAPL thickness to thickness of DNAPL is measured in a monitoring well.

3.8.3. Horizontal Movement in Saturated Zone

- Difficult to estimate potential for movement
- Water drive will produce fingery
  Classical fingery
  1) Due to instability
  2) Accentuated by heterogeneity
    - heterogeneity has overriding effect in most shallow aquifers.

Gradient required to move DNAPL (horizontally)

\[ \nabla p = \frac{25}{L_0 \left( \frac{R_e}{R_p} \right)^4} \]

\( L_0 = \) length of continuous DNAPL phase
\( R_e = \) throat
\( R_p = \) pore radii

eg. to determine radius of influence of capture well:

\[ \frac{dp}{dr} = \frac{1}{r} \frac{dh}{dr} \]

Difficulty in estimating \( L_0 \).
3.9 **MONITORING LNAPL & DNAPL**

**Screen for LNAPL**
- Screen perforations large to allow penetration
- Sump for DNAPL - sample with bailer

**DNAPL** - Bottom loading bailer
**LNAPL** - Top loading bailer scooping from upper surface

- Allow fluids to separate in sample \(\rightarrow\) decant.
- Partial emulsion induced by sampling process

Fractured bedrock sampling most difficult:
- Reduced DNAPL locations
- Infer source
- Drilling may remobilize "free" product
  by diluting fractures:

plume (dissolution)
FIGURE 5.25 Distribution of a dense nonaqueous phase liquid in the vadose and saturated zone.

FIGURE 5.26 Sloping interface between a static layer of DNAPL and flowing groundwater.

FIGURE 5.27 Movement of a DNAPL into a fractured bedrock aquifer that underlies a sand and gravel aquifer.
4. UNSATURATED FLOW

- Contaminants traverse vadose zone
  - Restricted (deliberative application)
  - Immunogenic
  - Dissolved

- Clay mineralogy
  - Reactive clays in many soils (clay = 2 mm)
  - Weathering products: Aluminum, Silica, Oxygen
  - Kaolinite (china clay) 5-20 m²/g
  - Montmorillonite - high surface area 700 m²/g

- Why important:
  - Cations
    (Net +ve charge @ ends)

  - Electrical double layer captures colloids (unbalanced + ve surface)
    attracts +ve cations
    => good buffers or attenuators

  - Electrostatic double layer affects hydrology
    Swelling clays swell and seal pathways.
    Water salinity ↑ → swelling → ↓ conductivity
    Process reversed if flushed by fresh water.
**STATIC SYSTEMS**

\[ P_c = P_{nw} - P_w \]

**Vadose Zone**

\[ P_c = P_{air} - P_{water} \]

**NAPL in Groundwater**

\[ P_c = P_{NAPL} - P_{water} \]

\[ \frac{dy}{dx} = -k \left( \frac{dh}{dx} \right) = -k_r \frac{k}{\mu} \frac{d\mu}{dx} \]

\[ \frac{dy}{dx} = k_r \frac{k}{\mu} \left( \frac{dP}{dx} + \rho g \frac{dz}{dx} \right) \]
4.1 Flow of Water in the Unsaturated Zone

- Neglect osmotic pressures
  (flow due to concentration gradients)
- Neglect electrochemical processes

\[ h = \psi(\theta) + z + \frac{v^2}{2g} \]

- \( h = \) total head
- \( z = \) elevation head
- \( \psi(\theta) = \) matrix potential
  i.e. matric pressure
- \( \theta = \) moisture content

\[ \psi(\theta) = \frac{P(\theta)}{Y_w} \]

Matric pressure, \( P(\theta) \) is a unique function of moisture content, \( \theta \).

Two popular relations (empirical) for \( P(\theta) \)-vs- \( \theta \)

1. Brooks and Corey (1966)
2. van Genuchten (1980).
4.1.1. Brooks & Corey (1965)

May define for $P_e - vs - \theta$ or $P_e - vs - S_w$:

$$\theta = \Theta_w + (\Theta_s - \Theta_w)(\frac{P}{P_b})^{-\lambda}$$

\[\left\{\begin{array}{l}
\Theta_w = \text{irreducible water content} \\
\Theta_s = \text{saturated water content} \\
P_b = \text{bubbling pressure} \\
P = \text{capillary pressure}
\end{array}\right.\]

$$\frac{\Theta - \Theta_w}{\Theta_s} + \left(1 - \frac{\Theta_w}{\Theta_s}\right) \left(\frac{P}{P_b}\right)^{-\lambda} \Rightarrow S = \Theta_w + \left(1 - \frac{\Theta_w}{\Theta_s}\right) \left(\frac{P}{P_b}\right)^{-\lambda}$$

$$\frac{S - \Theta_w}{1 - \Theta_w} = \left(\frac{P}{P_b}\right)^{-\lambda}$$

This is an empirical relation:

$$S_e = \left(\frac{P}{P_b}\right)^{-\lambda}$$

$$\log S_e = -\lambda \log \left(\frac{P}{P_b}\right)$$

$$\log S_e = -\lambda \left[\log P - \log P_b\right]$$

**Limiting conditions:**

For $P = P_b$ then $\log S_e = 0$; $S_e = 1$
Figure 4.5: Capillary pressure head as a function of effective saturation for porous materials with various pore sizes. (a) Plotted on arithmetic paper and (b) plotted on log-log paper. Source: R. H. Brooks and A. T. Corey, Proceedings, American Society of Civil Engineers, Irrigation and Drainage Division 92, no. 182 (1966): 61–87.

Behavior described by two parameters:

1) $P_h$ - Bubbling pressure

2) $\lambda$ - Slope of graph over 1 log cycle
4.1.2 von GENUCHTEN (1980) Empirical


\[ \Theta = \Theta_r + \frac{(\Theta_s - \Theta_r)}{[1 + (\omega \frac{P}{\Theta_r})^n]^m} \]

\[ \Theta_e = \frac{1}{[1 + (\omega \frac{P}{\Theta_r})^n]^m} \]

Parameters: \( m, n, \omega \)

\[ S = S_w + \frac{(1 - S_w)}{[1 + (\omega \frac{P}{S_w})^n]^m} \]

\[ S_e = \frac{1}{[1 + (\omega \frac{P}{S_w})^n]^m} \]

\[ \log \left( \frac{P_0}{\Theta_r} \right) \]

\[ \log \left( \frac{P_0}{S_w} \right) \]

water content

water saturation

\[ \alpha = \frac{S_w}{P_0} \left( 2^{1/m} - 1 \right)^{(1-m)} \]

\[ \alpha = \frac{1}{h_e} = \frac{S_w}{P_0} \]

not porous

\[ n = \frac{1}{1-m} \]
DETERMINING PARAMETERS

MEAN MOISTURE CONTENT

\[ \Theta_p = \frac{1}{2}(\Theta_s + \Theta_r) \]
\[ = \frac{1}{2}(0.5 + 0.1) = 0.3 \]

SLOPE AT \( \Theta_p \)

\[ f = 0.34 \]

DIMENSIONLESS SLOPE

\[ f_p = \frac{f}{\Theta_s - \Theta_r} \]
\[ = \frac{0.34}{0.5 - 0.1} = 0.85 \]

\[ f_p = \begin{cases} 
1 - \exp(-0.8f_p) & 0 \leq f_p \leq 1 \\
1 - 0.5755 + \frac{0.1}{f_p} + \frac{0.025}{f_p^2} & f_p > 1
\end{cases} \]

\[ m = \frac{1}{1 - m} = \frac{1}{1 - 0.5} = 2 \]

\[ \alpha = \frac{\gamma_w}{P_b} \left( 2^{1/m} - 1 \right)^{1-m} = \frac{\gamma_w}{P_b} (1.53) = \frac{1}{P_b/\gamma_w} \]
4.2 Flow of Water in the Unsaturated Zone

4.2.1 Hydraulic Conductivity

\[ K(\theta) = k_r(\theta) \frac{k}{\mu} \rho \frac{g}{m} \]

\[ k = \text{intrinsic permeability} \]

\[ k_r = \text{relative permeability} \]

van Genuchten (1980)

Empirically relate \( k_r \) to \( \theta \)

\[ K(\theta) = K_{sat} \left( \frac{S_e}{S_e^*} \right)^n \left[ 1 - \left( 1 - \frac{S_e}{S_e^*} \right)^m \right]^2 \]

\[ S_e = \frac{(\theta - \theta_r)}{(\theta_s - \theta_r)} \]

\[ K(h) = K_{sat} \frac{\left[ 1 - (\alpha h)^{n-1} \left[ 1 + (\alpha h)^n \right]^m \right]^2}{\left[ 1 + (\alpha h)^n \right]^{m/2}} \]

Note: \( h = \frac{P}{\gamma} \) and \( P \) related to \( S_e \) or \( \theta \)

\[ h \approx \frac{P}{\gamma} \]

Useful in complex numerical models for unsaturated flow...
Ideally, the relative conductivity, moisture content, and water capacity curves are determined directly by performing a series of tests on the soils involved in the study. However, in many cases they can be approximated using a set of measured or approximated constants and a set of empirical relationships. For example, one option for generating the curves is to use the van Genuchten functions (van Genuchten, 1980). The van Genuchten relationships are:

\[
K_r = \theta_e^{0.5}\left[1+(\alpha h)^{1/2}\right]^{2}
\]

and

\[
S_e = \theta_e = \left[1+\left(\alpha h\right)^{1/2}\right]^{-\gamma} \quad \text{for } h < 0
\]

\[
\theta_e = 1 \quad \text{for } h \geq 0
\]

where:

\[
\theta_s = \theta_r + \theta_e (\theta_i - \theta_r)
\]

\[
\gamma = 1 - \frac{1}{\beta}
\]

and

\[
\theta_s = \text{moisture content (dimensionless)}
\]

\[
S_e = \theta_e = \text{effective moisture content (dimensionless)}
\]

\[
\theta_i = \text{saturation moisture content (dimensionless)}
\]

\[
\theta_r = \text{residual moisture content (dimensionless)}
\]

\[
\beta, \gamma = \text{soil-specific exponents (dimensionless)}
\]

\[
\alpha = \text{soil-specific coefficient}
\]

\[
S_e = \theta_e = \left(\frac{\theta_m - \theta_r}{\theta_i - \theta_r}\right)
\]

\[
\gamma = m
\]

\[
\beta = n
\]

**Free parameters**

- \(\alpha\) represents \(1/(\theta_i \theta_m)\)
- \(\beta\) represents slope of \(P_e - vs - S_e\)

Table 5.1 lists a set of saturated and residual moisture contents and the van Genuchten \(\alpha\) and \(\beta\) terms for a variety of common soil types. When applying the \(\alpha\) term, care should be taken to convert it to the proper units.

<table>
<thead>
<tr>
<th>Soil Type</th>
<th>Saturated Moisture Content, (\theta_n)</th>
<th>Residual Moisture Content, (\theta_r)</th>
<th>(\alpha) [cm(^{-1})]</th>
<th>(\beta) [n]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clay**</td>
<td>0.38</td>
<td>0.068</td>
<td>0.008</td>
<td>1.09</td>
</tr>
<tr>
<td>Clay Loam</td>
<td>0.41</td>
<td>0.096</td>
<td>0.019</td>
<td>1.31</td>
</tr>
<tr>
<td>Loam</td>
<td>0.43</td>
<td>0.078</td>
<td>0.036</td>
<td>1.56</td>
</tr>
<tr>
<td>Loam Sand</td>
<td>0.41</td>
<td>0.057</td>
<td>0.124</td>
<td>2.28</td>
</tr>
<tr>
<td>Silt</td>
<td>0.46</td>
<td>0.034</td>
<td>0.106</td>
<td>1.37</td>
</tr>
<tr>
<td>Silt Loam</td>
<td>0.45</td>
<td>0.067</td>
<td>0.020</td>
<td>1.41</td>
</tr>
<tr>
<td>Silty Clay</td>
<td>0.35</td>
<td>0.070</td>
<td>0.005</td>
<td>1.09</td>
</tr>
<tr>
<td>Silty Clay Loam</td>
<td>0.43</td>
<td>0.069</td>
<td>0.010</td>
<td>1.23</td>
</tr>
<tr>
<td>Sandy</td>
<td>0.43</td>
<td>0.048</td>
<td>0.145</td>
<td>2.66</td>
</tr>
<tr>
<td>Sandy Clay</td>
<td>0.38</td>
<td>0.100</td>
<td>0.027</td>
<td>1.23</td>
</tr>
<tr>
<td>Sandy Clay Loam</td>
<td>0.39</td>
<td>0.100</td>
<td>0.059</td>
<td>1.48</td>
</tr>
<tr>
<td>Sandy Loam</td>
<td>0.41</td>
<td>0.066</td>
<td>0.075</td>
<td>1.89</td>
</tr>
</tbody>
</table>

**Agricultural soil, less than 60% clay**

Source: Carsel and Parrish (1988)