POROMECHANICS OF POROUS AND FRACTURED RESERVOIRS

Jishan Liu, Derek Elsworth

KIGAM, Daejeon, Korea

May 15-18, 2017

1. Poromechanics – Flow Properties (Jishan Liu)

1:1 Reservoir Pressure System – How to calculate overburden stress and reservoir pressure **Day 1**[1]
1:2 Darcy's Law – Permeability and its changes, reservoir classification
1:3 Mass Conservation Law – flow equations
1:4 Steady-State Behaviors – Solutions of simple flow problems
1:5 Hydraulic Diffusivity – Definition, physical meaning, and its application in reservoirs
1:6 Rock Properties – Their Dependence on Stress Conditions **Day 1**

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2. Poromechanics – Fluid Storage Properties (Jishan Liu)

2:1 Fluid Properties – How they change and affect flow **Day 2**
2:2 Mechanisms of Liquid Production or Injection
2:3 Estimation of Original Hydrocarbons in Place
2:4 Estimation of Ultimate Recovery or Injection of Hydrocarbons
2:5 Flow – Deformation Coupling in Coal
2:6 Flow – Deformation Coupling in Shale **Day 2**

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3. Poromechanics –Modeling Porous Medium Flows (Derek Elsworth)

3:1 Single porosity flows - Finite Element Methods [2:1] Lecture **Day 3**
3:2 2D Triangular Constant Gradient Elements [2:3] Lecture
3:3 Transient Behavior - Mass Matrices [2:6] Lecture
3:4 Transient Behavior - Integration in Time [2:7] Lecture
3:5 Dual-Porosity-Dual-Permeability Models [6:1] Lecture **Day 3**

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4. Poromechanics –Modeling Coupled Porous Medium Flow and Deformation (Derek Elsworth)

4:1 Mechanical properties – http://www.ems.psu.edu/~elsworth/courses/geoee500/GeoEE500_1.PDF **Day 4**
4:2 Biot consolidation – http://www.ems.psu.edu/~elsworth/courses/geoee500/GeoEE500_1.PDF
4:3 Dual-porosity poroelasticity
4:4 Mechanical deformation - 1D and 2D Elements [5:1][5:2] Lecture
4:5 Coupled Hydro-Mechanical Models [6:2] Lecture **Day 4**

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[1] *All sessions nominally1h:15m*

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[2] *All sessions nominally1h:15m*

Summary of Notation – Diffusion Equation

Tensor: $A\dfrac{\partial c}{\partial t} + \nabla\cdot(-D\nabla c) = R \quad \text{with} \quad \nabla = \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} \quad and \quad \nabla\cdot\nabla = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

Matrix: $A\dot{c} - \underline{\nabla}^T D\ \underline{\nabla} c = R \quad \text{with} \quad \underline{\nabla} = \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} \quad and \quad \nabla\cdot\nabla = \underline{\nabla}^T\underline{\nabla} = \nabla^2$

Indicial:

$$A\frac{\partial c}{\partial t} - D(\frac{\partial^2 c}{\partial {x_1}^2} + \frac{\partial^2 c}{\partial {x_2}^2} + \frac{\partial^2 c}{\partial {x_3}^2}) = R \quad or \quad A\frac{\partial c}{\partial t} - D(\frac{\partial^2 c}{\partial {x_i}^2}) = R \quad or \quad A\dot{c} - Dc_{i,ii} = R \quad or \quad A\frac{\partial c}{\partial t} - Ddiv(gradc) = R$$

Epanded: $A\dfrac{\partial c}{\partial t} - D(\dfrac{\partial^2 c}{\partial x^2} + \dfrac{\partial^2 c}{\partial x^2} + \dfrac{\partial^2 c}{\partial y^2}) = R$

Advective-Diffusive Flows

$$A\frac{\partial c}{\partial t} + \nabla\cdot(-D\nabla c) = R - \mathbf{v}\cdot\nabla c$$

Momentum Transfer - Fluid Mechanics – Navier-Stokes Equations (Incompressible)

Tensor:

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{F} - \nabla P + \mu\nabla^2\mathbf{v} \quad \text{with} \quad \nabla^2 = \nabla\cdot\nabla = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

$$\nabla\cdot\mathbf{v} = 0$$

Expanded:

$$\rho\frac{\partial}{\partial t}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + \rho[v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}]\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} - \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} P + \mu[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}]\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}] = 0$$

3:1 Single porosity flows - Finite Element Methods [2:1]

https://youtu.be/46QiOhaC47c

[2:1] Fluid Flow and Pressure Diffusion

Recap of FEM

Comsol Applied to Flow

1D Element

PROCESS COUPLINGS [T-H-M-C]

$$\begin{array}{c}\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \overset{R_{22}}{\cdots} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \\ \text{Conductance}\end{array} \begin{Bmatrix} \underline{u} \\ \underline{p} \\ \underline{T} \\ \underline{c} \end{Bmatrix} + \begin{array}{c}\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \overset{S_{22}}{\cdots} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \\ \text{Storage}\end{array} \begin{Bmatrix} \dot{\underline{u}} \\ \dot{\underline{p}} \\ \dot{\underline{T}} \\ \dot{\underline{c}} \end{Bmatrix} = \begin{Bmatrix} \dot{\underline{f}} + \ldots \\ \underline{q}_F + \ldots \\ \underline{q}_T + \ldots \\ \underline{q}_M + \ldots \end{Bmatrix}$$

<u>Need to Understand:</u>

① R_{22} $\qquad \underline{q}_F = \underline{R}_{22}\underline{p}$ or $q_F = \underline{\bar{R}}_{22}\underline{h}$

$R_{22} = \int_V \underline{a}^T \underline{D}\, \underline{a}\, dV$

Conductance matrix $\begin{cases} 1\text{-D} \\ 2\text{-D} \end{cases}$

② S_{22} $\qquad$ Form of S_{22} – $S_{22} = S_s \int \underline{b}^T \underline{b}\, dV$

③ Transient behavior:

$$q_F = R_{22}\underline{p} + S_{22}\dot{\underline{p}}$$

Time stepping:

Implicit.

Explicit.

EQUIVALENCE OF: HYDRAULIC CONDUCTIVITY (K) & PERMEABILITY (k)

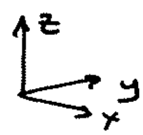
and

HEAD (h) AND PRESSURE (p).

$$\frac{K}{\rho g} = \frac{k}{\mu} \qquad (1)$$

$$h = \frac{p}{\rho g} + z \qquad (2)$$

$$q_i = -\frac{k}{\mu}\left(\frac{\partial p}{\partial x_i} + \rho g \frac{\partial z}{\partial x_i}\right) \qquad (3)$$



Take derivative of (h) and multiply by ρg:

$$\rho g \frac{\partial}{\partial x}(h) = \cancel{\rho g}\frac{1}{\cancel{\rho g}}\frac{\partial p}{\partial x} + \rho g \frac{\partial z}{\partial x} \qquad (4)$$

Substitute (4) into (3)

$$q_i = -\underbrace{\frac{k}{\mu}(\rho g)}_{K}\frac{\partial h}{\partial x} = -K\frac{\partial h}{\partial x} \qquad (5)$$

SYSTEM TYPES

SOLID MECHANICS	FLOW SYSTEM	TRANSPORT
Conservation of momentum: (Equilibrium), $VW_I = VW_E$	Conservation of mass: $\underline{\nabla}^T \underline{q} = 0$	Conservation of mass $\underline{\nabla}^T \underline{q} = 0$
Continuity (Compatibility): $\underline{\varepsilon} = \underline{a}\,\underline{u}$	Continuity: $\underline{h}_{,} = \underline{a}\,\underline{h}$	Continuity: $\underline{c}_{,} = \underline{a}\,\underline{c}$
Constitutive relation: $\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$	Constitutive reln. $\underline{v} = \underline{D}\,\underline{h}_{,}$	Constitutive: diffusion - $\underline{v}_1 = \underline{D}\,\underline{c}_{,}$ advective - $\underline{v}_2 = \underline{A}\,\underline{c}$
Initial Conditions	ICs	ICs
Boundary Conditions	BCs	BCs

— SOLVE SYSTEM EQUATIONS —

MASS BALANCE – FLOW

M_a = MASS RATE IN – MASS RATE OUT

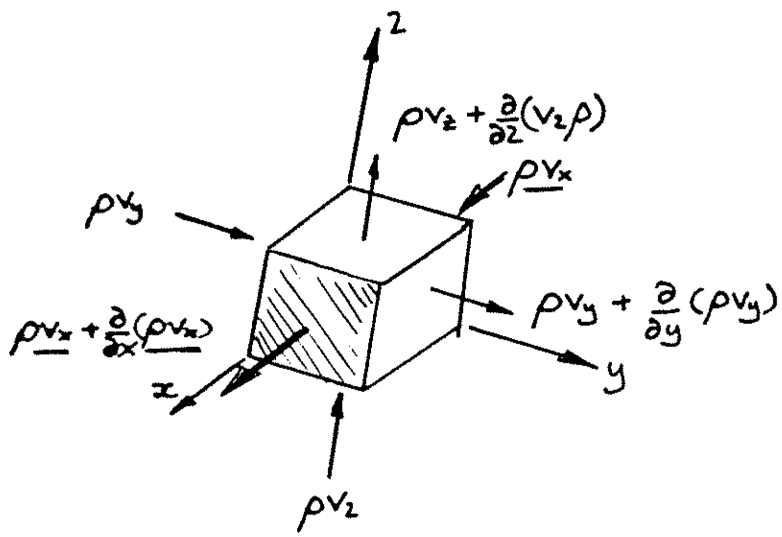


Figure 2.2.1 Unit differential cube, dx = dy = dz = 1

For x-direction: $\rho v_x - [\rho v_x + \frac{\partial}{\partial x}(\rho v_x)] = M_a$

gives $-\frac{\partial}{\partial x}(\rho v_x) - \frac{\partial}{\partial y}(\rho v_y) - \frac{\partial}{\partial z}(\rho v_z) = M_a$

Assume ρ const. then $\rho [-\frac{\partial}{\partial x}(v_x) - \frac{\partial}{\partial y}(v_y) - \frac{\partial}{\partial z}(v_z)] = \rho \underset{1}{S_s} \frac{\partial h}{\partial t}$

Continuity equation.

Specific storage

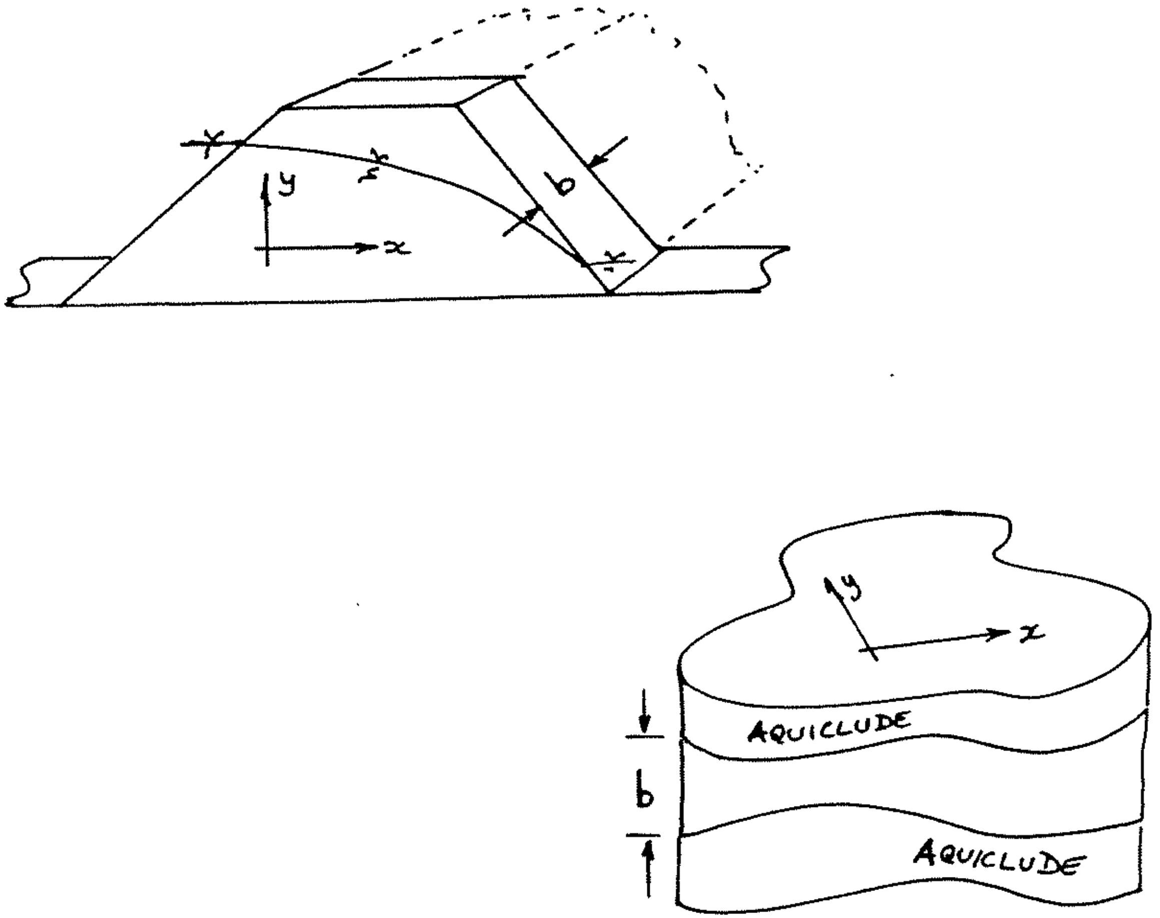


Figure 2.2.2. Two dimensional geometries (a) Confined vertical section; (b) Confined flow in a horizontal aquifer.

Darcy's Law $\quad V_x = -K_x \frac{\partial h}{\partial x}$ etc.

Substitute into continuity equation

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = S_s \frac{\partial h}{\partial t}$$

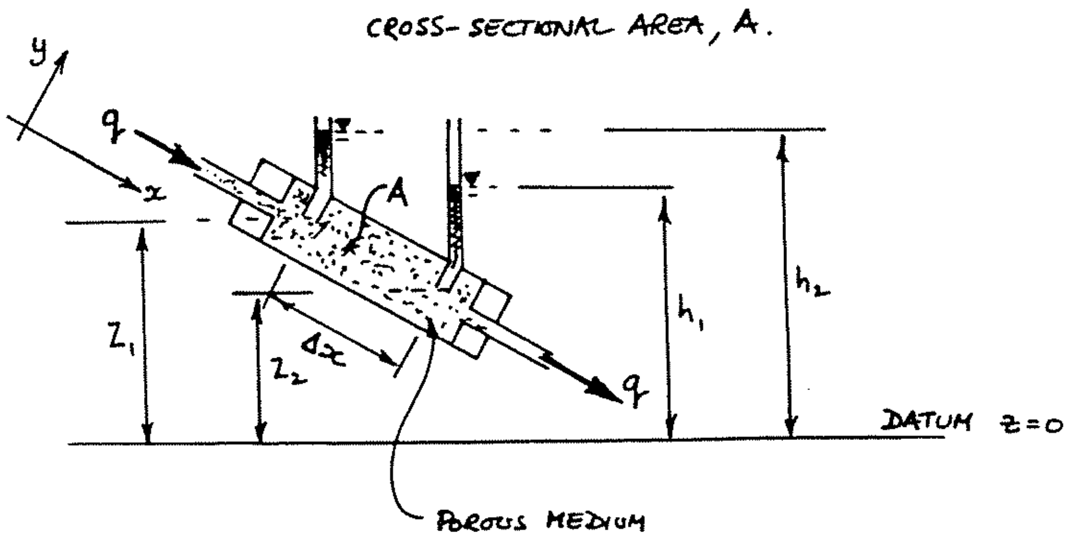


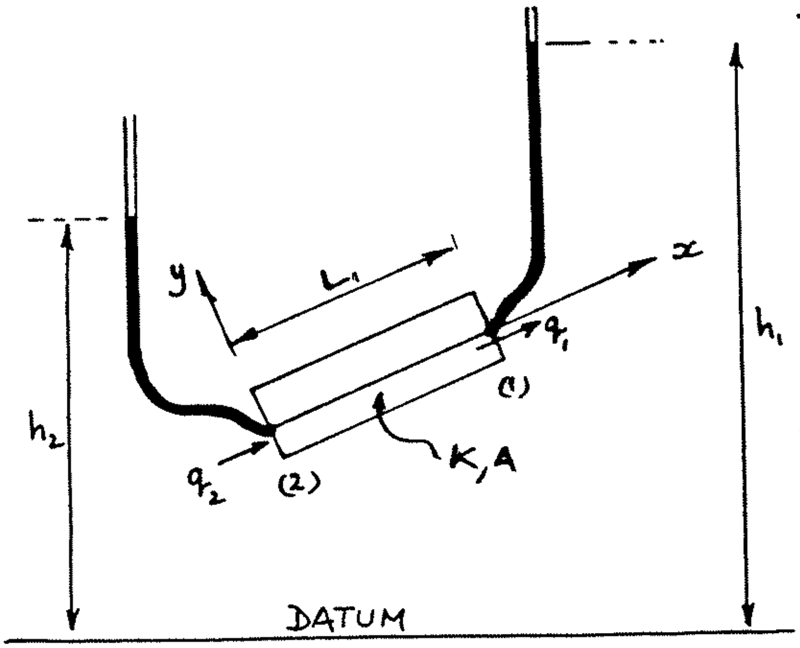
Figure 2.1.2.1 Constant head permeameter as defined by D'Arcy's experiment

Other variants of equation:

(2-D) $\quad K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} = S_s \frac{\partial h}{\partial t}$

(Aerial 2-D) $\quad T = Kb \; ; \; S = S_s b$

(Areal) $\quad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t}$



Darcy's Law

$$q_1 = +A_1 K_1 \frac{(h_2 - h_1)}{L_1}$$

$$q_2 = -q_1$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = -\frac{A_1 K_1}{L_1} \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{} \begin{Bmatrix} h_1 \\ h_2 \end{Bmatrix}$$

$$\underline{q} = \underline{K} \quad \underline{h}$$

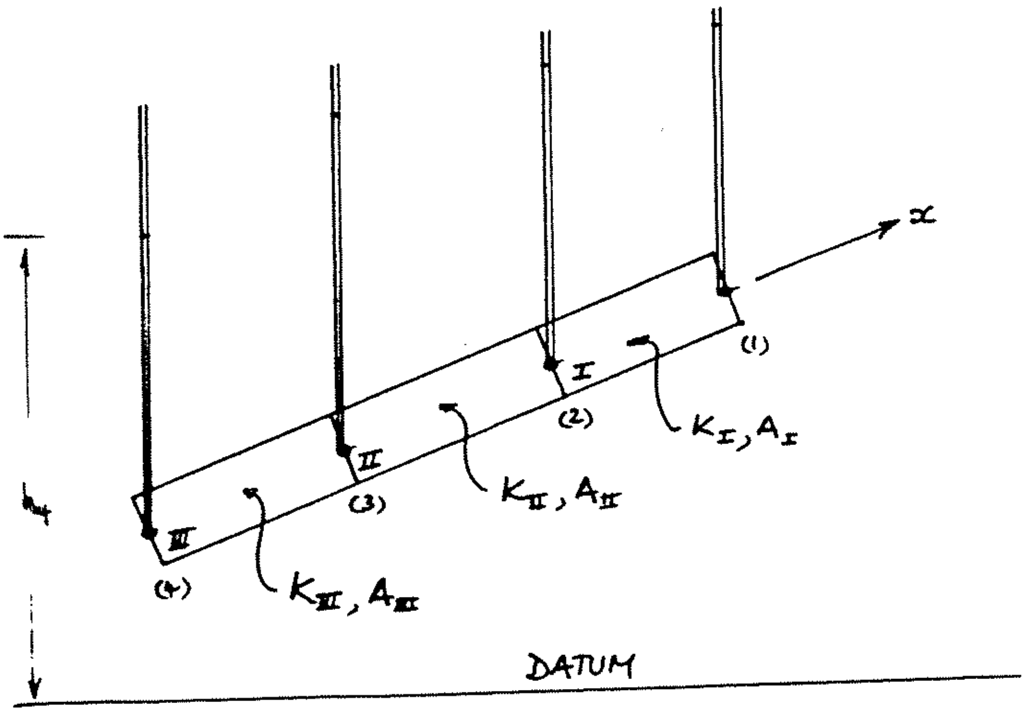


Figure 2.3.1 (a) Single element representing flow in a pipe; (b) Multiple elements joined in series

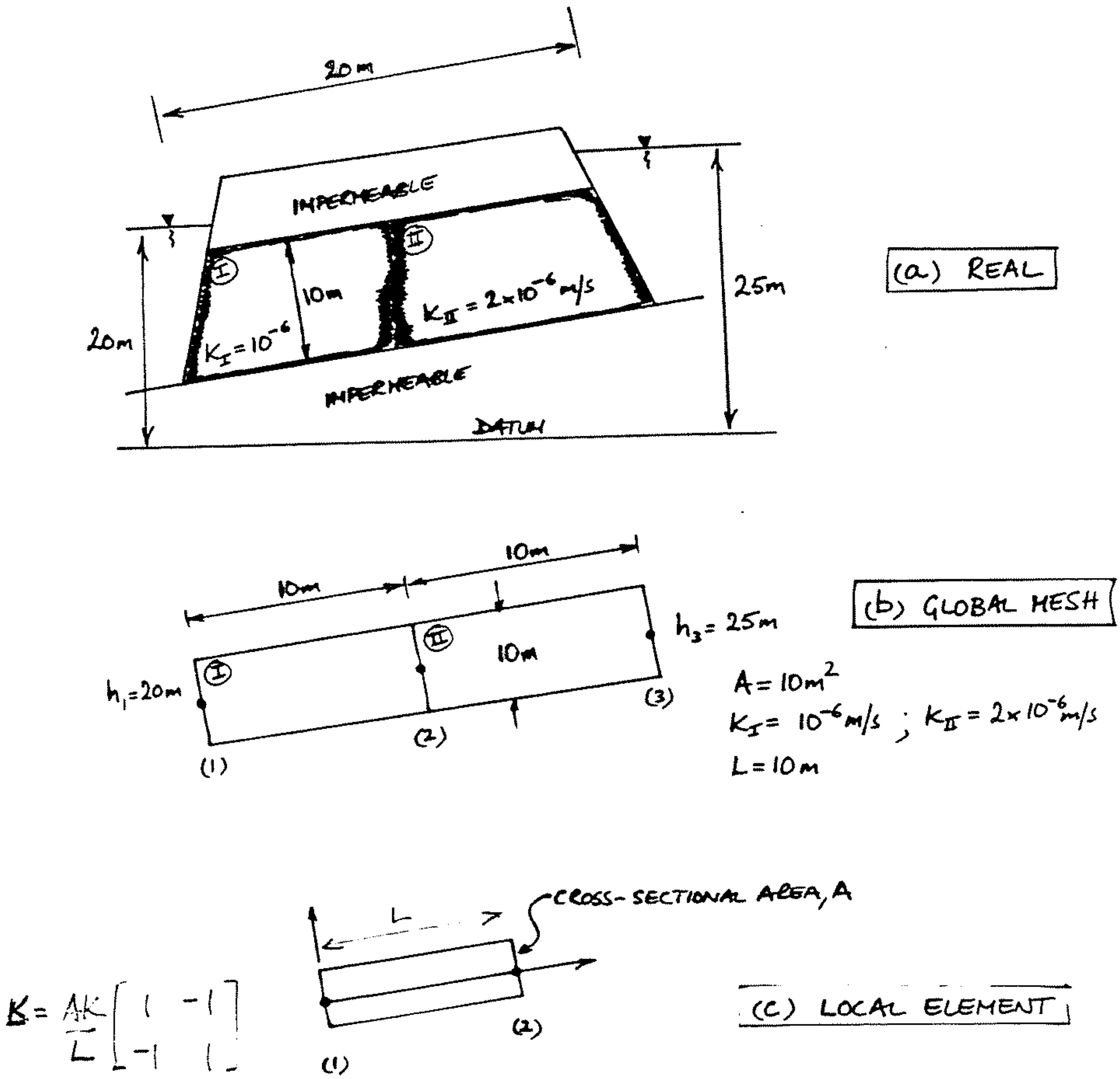


Figure 2.3.2 Illustrative example for one dimensional flow

ONE DIMENSIONAL EXAMPLE

$$\left.\begin{array}{l} A = 10\,m^2 \\ K = 10^{-6}\ m/s \\ L = 10\,m \end{array}\right\} \frac{AK}{L} = 10^{-6}\ (m^2/s)$$

$$K_I = 10^{-6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad K_{II} = 2\times 10^{-6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = 10^{-6}\begin{bmatrix} 1 & -1 & 0 \\ -1 & (1+2) & -2 \\ 0 & -2 & 2 \end{bmatrix}\begin{Bmatrix} h_1 = 20 \\ h_2 = ? \\ h_3 = 25 \end{Bmatrix}$$

Solve: $q_2 = 0$ since no prescribed flux

solve central equation only !!

$$\cancel{q_2}^{0} + 20\times 10^{-6} + 2\times 25\times 10^{-6} = 3\times 10^{-6}\times h_2$$

$$\frac{70}{3} = h_2$$

Resubstitute to determine flux magnitude:

$$q_1 = \left(20 - \frac{70}{3}\right)\times 10^{-6} = -3.3\dot{3}\times 10^{-6}\ m^3/s$$

$$q_2 = 0$$

$$q_3 = \left(-2\left(\frac{70}{3}\right) + 2(25)\right)\times 10^{-6} = +3.33\times 10^{-6}\ m^3/s$$

Fluxes sum to zero!

3:2 2D Triangular Constant Gradient Elements [2:3]

http://youtu.be/a_8VZejdgTA

[2:3] Fluid Flow and Pressure Diffusion

Recap

2D Triangular (Constant Gradient) Elements

Derivation

Example

EGEEfem

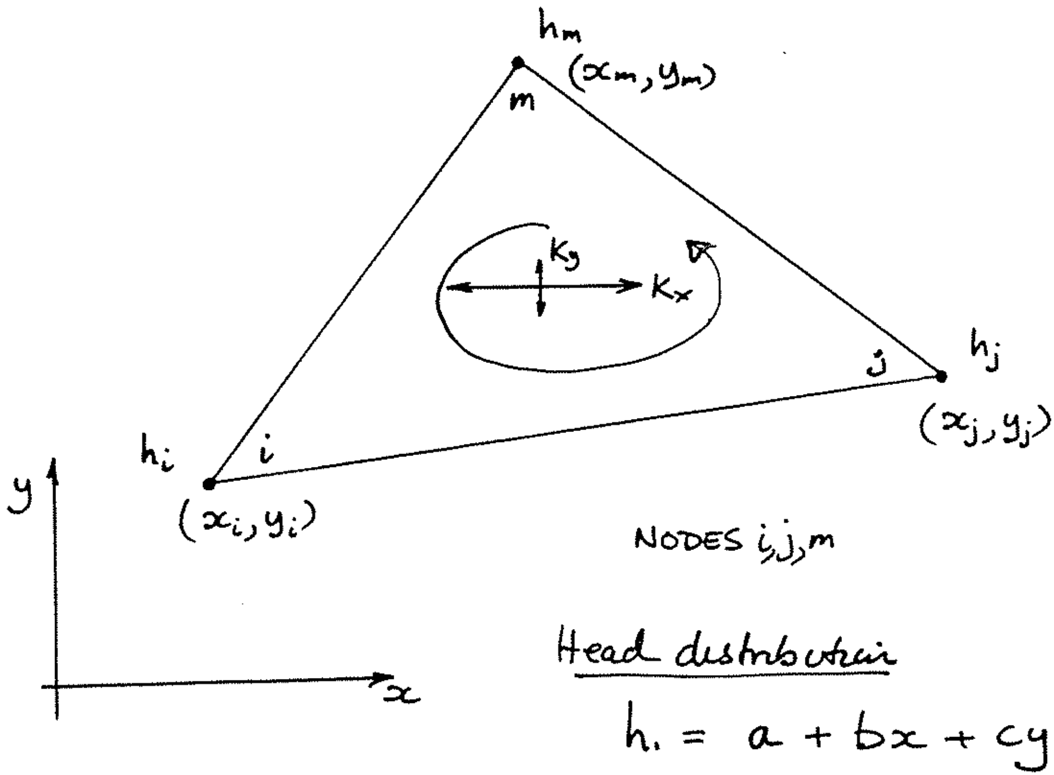


Figure 2.4.2.1 Geometry of a triangular element

TRIANGULAR ELEMENT

$$\underline{V} = \underline{D}\,\underline{h}_,$$

$$\underline{K} = \int_V \underline{a}^T \underline{D}\,\underline{a}\; dV \qquad (1)$$

$$\underline{D} = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \qquad K_{xy} = K_{yx} \qquad \underline{or} \qquad \begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \begin{Bmatrix} \partial h/\partial x \\ \partial h/\partial y \end{Bmatrix}$$

'a' Matrix $\qquad \underline{h}_, = \underline{a}\,\underline{h} \qquad$ or $\qquad \begin{Bmatrix} \partial h/\partial x \\ \partial h/\partial y \end{Bmatrix} = \underline{a}\,\underline{h} \qquad (2)$

Choose shape functions: $\qquad h = a + bx + cy \qquad (3)$

$$\therefore \left.\begin{aligned} \frac{\partial h}{\partial x} &= b \\ \frac{\partial h}{\partial y} &= c \end{aligned}\right\} (4)$$

The head magnitudes are defined as h_i, h_j, h_m at the nodes.
∴ 3 equations may be determined as:

$$\left.\begin{aligned} h_i &= a + bx_i + cy_i \\ h_j &= a + bx_j + cy_j \\ h_m &= a + bx_m + cy_m \end{aligned}\right\} (5)$$

Writing in matrix form; the equations (5) may be presented as

$$\begin{Bmatrix} h_i \\ h_j \\ h_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \qquad (6)$$

Inverting (6) to give the coefficients (b) and (c) of equation (4) yields:

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} h_i \\ h_j \\ h_m \end{Bmatrix} \qquad (7)$$

2Δ = the determinant of the equation $\quad (2\Delta = A_{11} + x_i A_{21} + y_i A_{31})$

Δ = area of triangle.

Returning to (2) and (4) and using (7), then

$$\begin{Bmatrix} \partial h/\partial x \\ \partial h/\partial y \end{Bmatrix} = \begin{Bmatrix} b \\ c \end{Bmatrix} = \underbrace{\frac{1}{2\Delta} \begin{bmatrix} A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{\underline{a}} \begin{Bmatrix} h_i \\ h_j \\ h_m \end{Bmatrix} \qquad (8)$$

$\underline{K} = \int_V \underline{a}^T \underline{D}\, \underline{a}\, dV$

Since $\underline{a}$ and $\underline{D}$ are constant over the element, they are removed from the integral.

$$\underline{K} = \int_V \underline{a}^T \underline{D}\, \underline{a}\, dV = \underline{a}^T \underline{D}\, \underline{a} \int_V dV = \underline{a}^T \underline{D}\, \underline{a}\, \Delta \text{ thickness}$$

where Δ is the scalar magnitude of element area.

STANDARD RESULT

$$\begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix}}_{\underline{B}} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \frac{1}{|\underline{B}|} \begin{bmatrix} (x_j y_m - x_m y_j) & (x_m y_i - x_i y_m) & (x_i y_j - x_j y_i) \\ (y_j - y_m) & (y_m - y_i) & (y_i - y_j) \\ (x_m - x_j) & (x_i - x_m) & (x_j - x_i) \end{bmatrix} \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix}$$

$$|\underline{B}| = 2\Delta = 1 . \begin{vmatrix} x_j & y_j \\ x_m & y_m \end{vmatrix} - x_i . \begin{vmatrix} 1 & y_j \\ 1 & y_m \end{vmatrix} + y_i \begin{vmatrix} 1 & x_j \\ 1 & x_m \end{vmatrix}$$

$$2\Delta = (x_j y_m - x_m y_j) - x_i (y_m - y_j) + y_i (x_m - x_j)$$

II.5 Inversion (Adjoint Matrix)

It can be shown that

$$\mathbf{a}\,(\mathrm{adj}\ \mathbf{a}) = |\mathbf{a}|\,\mathbf{I} \tag{II.11}$$

where $|\mathbf{a}|$ is the determinant of the matrix $\mathbf{a}$ and adj $\mathbf{a}$, called the *adjoint* matrix, is the transpose of the matrix of cofactors of the determinant. Comparing (II.10) and (II.11) we see that

$$\mathbf{a}^{-1} = \frac{\mathrm{adj}\ \mathbf{a}}{|\mathbf{a}|} \tag{II.12}$$

from which it is clear that the inverse does not exist when $|\mathbf{a}|$ is zero, in which case $\mathbf{a}$ is said to be *singular*.

To illustrate the method we shall determine the inverse of the matrix

$$\mathbf{H} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \tag{II.13}$$

If we delete the pth row and qth column from the determinant of the matrix we obtain the minor H'_{pq}, e.g. deleting row 3 and column 1 we have

$$H'_{31} = \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \tag{II.14}$$

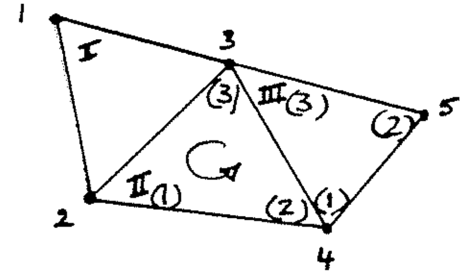
The cofactor $\bar{H}_{pq}$ is the product of the minor and $(-1)^{(p+q)}$. When the cofactors are written as a matrix and then transposed we have the adjoint matrix

$$\mathrm{adj}\ \mathbf{H} = \begin{bmatrix} \begin{vmatrix} x_j & y_j \\ x_m & y_m \end{vmatrix} & -\begin{vmatrix} x_i & y_i \\ x_m & y_m \end{vmatrix} & \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \\ -\begin{vmatrix} 1 & y_j \\ 1 & y_m \end{vmatrix} & \begin{vmatrix} 1 & y_i \\ 1 & y_m \end{vmatrix} & -\begin{vmatrix} 1 & y_i \\ 1 & y_j \end{vmatrix} \\ \begin{vmatrix} 1 & x_j \\ 1 & x_m \end{vmatrix} & -\begin{vmatrix} 1 & x_i \\ 1 & x_m \end{vmatrix} & \begin{vmatrix} 1 & x_i \\ 1 & x_j \end{vmatrix} \end{bmatrix} \tag{II.15}$$

For example H'_{31} of (II.14) is transposed to row 1 column 3. Expanding the determinants we have

$$\mathrm{adj}\ \mathbf{H} = \begin{bmatrix} (x_j y_m - x_m y_j) & -(x_i y_m - x_m y_i) & (x_i y_j - x_j y_i) \\ -(y_m - y_j) & (y_m - y_i) & -(y_j - y_i) \\ (x_m - x_j) & -(x_m - x_i) & (x_j - x_i) \end{bmatrix} \tag{II.16}$$

The inverse is obtained by dividing adj $\mathbf{H}$ by the determinant of $\mathbf{H}$.



LOCAL ELEMENT MATRICES $\underline{K}\ \underline{h} = \underline{q}$

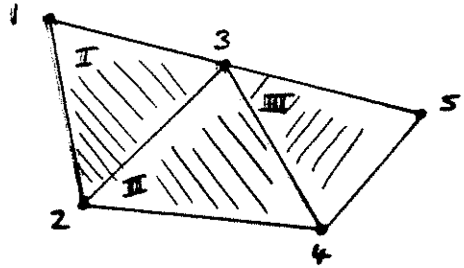
$$\begin{bmatrix} \mathrm{I}_{11} & \mathrm{I}_{12} & \mathrm{I}_{13} \\ \mathrm{I}_{21} & \mathrm{I}_{22} & \mathrm{I}_{23} \\ \mathrm{I}_{31} & \mathrm{I}_{32} & \mathrm{I}_{33} \end{bmatrix} \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \qquad \text{ELEMENT I}$$

$$\begin{array}{c c} & \begin{array}{ccc} 2 & 4 & 3 \end{array} \\ \begin{array}{c} 2 \\ 4 \\ 3 \end{array} & \begin{bmatrix} \mathrm{II}_{22} & \mathrm{II}_{24} & \mathrm{II}_{23} \\ \mathrm{II}_{42} & \mathrm{II}_{44} & \mathrm{II}_{43} \\ \mathrm{II}_{32} & \mathrm{II}_{34} & \mathrm{II}_{33} \end{bmatrix} \end{array} \begin{Bmatrix} h_2 \\ h_4 \\ h_3 \end{Bmatrix} = \begin{Bmatrix} q_2 \\ q_4 \\ q_3 \end{Bmatrix} \qquad \text{ELEMENT II}$$

$$\begin{bmatrix} \mathrm{III}_{44} & \mathrm{III}_{45} & \mathrm{III}_{43} \\ \mathrm{III}_{54} & \mathrm{III}_{55} & \mathrm{III}_{53} \\ \mathrm{III}_{34} & \mathrm{III}_{35} & \mathrm{III}_{33} \end{bmatrix} \begin{Bmatrix} h_4 \\ h_5 \\ h_3 \end{Bmatrix} = \begin{Bmatrix} q_4 \\ q_5 \\ q_3 \end{Bmatrix} \qquad \text{ELEMENT III}$$

NOTE : SUBSCRIPTS REPRESENT GLOBAL NODE NUMBERS

Figure 2.4.2.2 Local element conductance matrices for a three element system



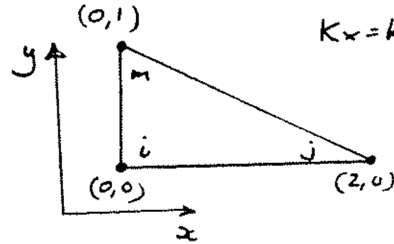
GLOBAL SYSTEM MATRIX $\underline{K}\,\underline{h} = \underline{q}$

$$
\underbrace{
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \mathrm{I}_{11} & \mathrm{I}_{12} & \mathrm{I}_{13} & & \\
2 & \mathrm{I}_{21} & \mathrm{I}_{22}+\mathrm{II}_{22} & \mathrm{I}_{23}+\mathrm{II}_{23} & \mathrm{II}_{24} & \\
3 & \mathrm{I}_{31} & \mathrm{I}_{23}+\mathrm{II}_{32} & \mathrm{I}_{33}+\mathrm{II}_{33}+\mathrm{III}_{33} & \mathrm{II}_{24}+\mathrm{III}_{34} & \mathrm{III}_{35} \\
4 & & \mathrm{II}_{42} & \mathrm{II}_{43}+\mathrm{III}_{43} & \mathrm{II}_{44}+\mathrm{III}_{34} & \mathrm{III}_{45} \\
5 & & & \mathrm{III}_{53} & \mathrm{III}_{54} & \mathrm{III}_{55}
\end{array}
}_{\underline{K}\,\text{global}}
\begin{Bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{Bmatrix}
=
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{Bmatrix}
$$

INDIVIDUAL TERMS ARE ADDED FROM THE LOCAL ELEMENT MATRICES.

Figure 2.4.2.3 Global assembly for three element example

$K_x = K_y = 1.0\ m/s$



Example 2.4.2.1

Evaluate the element conductance matrix for the triangular element shown.

From equation (2.4.2.9)

$$\underline{a} = \frac{1}{2\Delta}\begin{bmatrix} A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

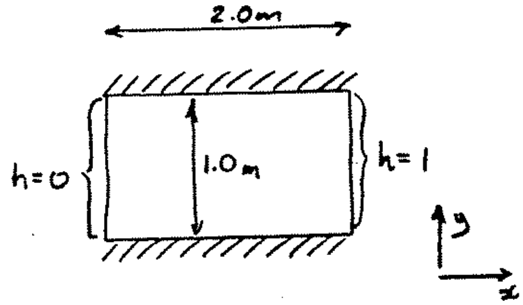
$$\underline{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

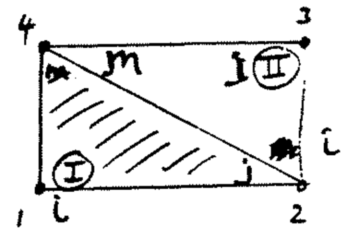
$$\underline{K} = \underline{a}^T \underline{D}\, \underline{a}\, \Delta = \frac{1}{2\Delta}\begin{bmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{2\Delta}\begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 2 \end{bmatrix}\Delta$$

$$\underline{K} = \frac{1}{4\Delta}\begin{bmatrix} 5 & -1 & -4 \\ -1 & 1 & 0 \\ -4 & 0 & 4 \end{bmatrix}$$ where $\Delta = 1.0$

Example 2.6.2.2

...r the system shown, evaluate flow rates using the finite element method.





$K_x = K_y = 1.0 \text{ m/s}$

Element Conductance Matrices $\quad \underline{K}\,\underline{h} = \underline{q}$

$$\underline{K}_I = \frac{1}{4}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 4 \end{array} \\ \begin{bmatrix} 5 & -1 & -4 \\ -1 & 1 & 0 \\ -4 & 0 & 4 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 4 \end{array} \quad ; \quad \underline{K}_{II} = \frac{1}{4}\begin{array}{c} \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 4 & -4 & 0 \\ -4 & 5 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array}$$

Note: $\underline{K}_{II}$ may be derived directly from $\underline{K}_I$ in this particular case since they possess identical geometries. Element II is merely rotated. The permutation $4 \rightarrow 2$; $1 \rightarrow 3$ and $2 \rightarrow 4$ may be used to exchange locations as, for example, $K_{I_{14}} = K_{II_{32}}$ as illustrated.

Global Conductance Matrix $\quad \underline{K}\,\underline{h} = \underline{q}$

$$\frac{1}{4}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 5 & -1 & & -4 \\ -1 & (1+4) & -4 & (0+0) \\ & -4 & 5 & -1 \\ -4 & (0+0) & -1 & (4+1) \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{Bmatrix} h_1 = 0 \\ h_2 = 1.0 \\ h_3 = 1.0 \\ h_4 = 0 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \frac{1}{4}\begin{Bmatrix} -1 \\ +1 \\ +1 \\ -1 \end{Bmatrix}$$

Note: Discharge at left side $= q_1 + q_4 = -\frac{1}{2}$ m³/s

Discharge at right side $= q_2 + q_3 = +\frac{1}{2}$ m³/s

Total accumulation $\quad 0$ m³/s

Check with D'Arcy's law $\quad q = AK\dfrac{dh}{dl} = (1.0)(1.0)\dfrac{1.0}{2.0} = \dfrac{1}{2}$ m³/s

Q.E.D.

3:3 Transient Behavior - Mass Matrices [2:6]

https://youtu.be/fl5AVltg51E

[2:6] Fluid Flow and Pressure Diffusion

Recap

Transient Behavior $\underline{K}\underline{h} + \underline{S}\dot{\underline{h}} = \underline{q}$

"Mass" matrices

TRANSIENT FLOW

General diffusion equation in FEM form is

$$\int_A \underline{a}^T \underline{D}\, \underline{a}\, dx\, dy\, \underline{h} - \int_A \underline{b}^T \underline{b}\, dx\, dy\, \underline{Q}$$

$$+ \underbrace{S_s \int_A \underline{b}^T \underline{b}\, dx\, dy}_{\underline{S}} \frac{\partial}{\partial t} \underline{h} = \underline{q} \qquad (1)$$

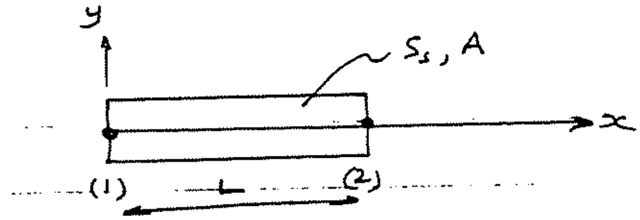
Use shorthand to represent equation as (neglecting $\underline{Q}$)

$$\underline{K}\, \underline{h}_t + \underline{S}\, \dot{\underline{h}}_t = \underline{q} \qquad (2)$$

Two steps in solution:

1) Determine $\underline{S}$ matrix
2) Perform integration of equations in time to remove $\partial/\partial t$.

Storage Matrix



$$\underline{S} = S_s \int_V b^T b \, dV$$

$$\underline{S} = A S_s \int_0^L \underline{b}^T \underline{b} \, dx \qquad \text{Shape functions: } b = [\,(1 - \tfrac{x}{L})\,;\ \tfrac{x}{L}\,]$$

Substituting into matrix relation:

$$\underline{S} = A S_s \int_0^L \begin{Bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{Bmatrix} \{1 - \tfrac{x}{L}\,;\ \tfrac{x}{L}\} \, dx = A S_s \int_0^L \begin{bmatrix} (1-\frac{x}{L})^2 & \frac{x}{L}(1-\frac{x}{L}) \\ \frac{x}{L}(1-\frac{x}{L}) & (1-\frac{x}{L})^2 \end{bmatrix} d$$

Evaluate component integrals:

$$\int_0^L (1 - \tfrac{x}{L})^2 \, dx = \int_0^L (1 - \tfrac{2x}{L} + \tfrac{x^2}{L^2}) \, dx = [\, x - \tfrac{x^2}{L} + \tfrac{x^3}{3L^2} \,]_0^L$$

$$= L - L + \tfrac{L}{3} = \tfrac{1}{3} L$$

$$\int_0^L \tfrac{x}{L}(1 - \tfrac{x}{L}) \, dx = \int_0^L (\tfrac{x}{L} - \tfrac{x^2}{L^2}) \, dx = [\, \tfrac{1}{2}\tfrac{x^2}{L} - \tfrac{1}{3}\tfrac{x^3}{L^2} \,]_0^L = \tfrac{1}{2}L - \tfrac{1}{3}L = \tfrac{1}{6}L$$

Resubstituting as Consistent Mass
$$\underline{S} = A S_s L \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$$

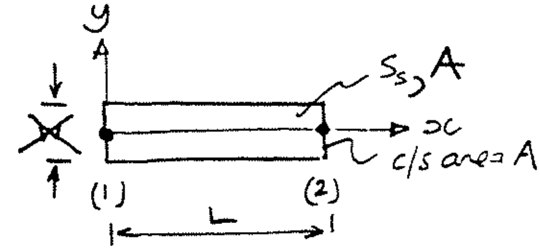
Lumped Mass
$$\underline{S} = A S_s L \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Note total mass of element is $\Sigma S_{ij} = 1 \underbrace{AL}_{V} S_s$

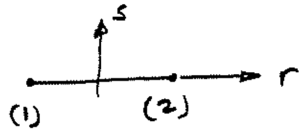
Example 2.4.4 Storage Matrix

Consider a one-dimensional element of specific storage, S_s, and length, L.

GLOBAL



LOCAL



Use the isoparametric concept

Then shape function, b. $\quad \underline{b} = \frac{1}{2}[(1-r) \,;\, (1+r)]$ (1)

Storage matrix

$$\underline{S} = A S_s \int_{0}^{L} \underline{b}^T \underline{b} \, dx \quad = \quad S_s A \int_{-1}^{+1} \underline{b}^T \underline{b} \, |\underline{J}| \, dr \qquad (2)$$

$$|\underline{J}| = \frac{\partial x}{\partial r} = \frac{L}{2} \qquad (3)$$

Resubstituting (1) and (3) into (2) gives

$$\underline{S} = \frac{S_s A L}{8} \int_{-1}^{+1} \underbrace{\begin{bmatrix} (1-r)^2 & (1-r)(1+r) \\ (1-r)(1+r) & (1+r)^2 \end{bmatrix}}_{\underline{b}^T \underline{b}} dr \qquad (4)$$

This may be evaluated either analytically or using 2 point quadrature $r=2$).

$$\underline{S} = \frac{S_s A L}{8} \begin{bmatrix} 2.6\dot{6} & 1.3\dot{3} \\ 1.3\dot{3} & 2.6\dot{6} \end{bmatrix} \quad \text{(Consistent)} \qquad (5)$$

This consistent matrix may be lumped at the nodes as

$$\underline{S} = \frac{S_s A L}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(Lumped)} \qquad (6)$$

Total amount of fluid in storage is given by $\sum_{i=1}^{2} \sum_{j=1}^{2} S_{ij} = S_s A L$

or $S_s \times$(Element volume).

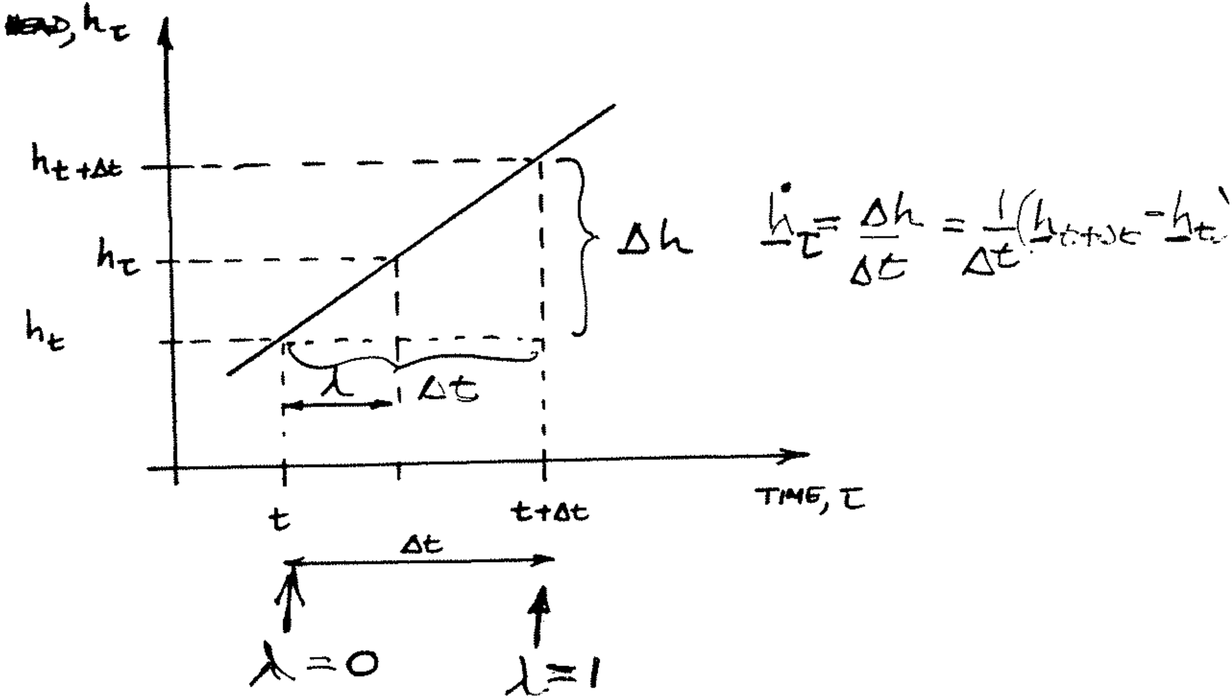


Figure 2.4.4.1.1 Variation in nodal head over the time interval t to t + Δt

IMPLICIT TIME INTEGRATION

Exact statement @ time τ $\quad \underline{K}\,\underline{h}_\tau + \underline{S}\,\underline{\dot{h}}_\tau = q_\tau \quad (1)$

Write FE equations at time $\tau = t + \Delta t$ $\therefore$ using (1)

$$\underline{K}\,\underline{h}_{t+\Delta t} + \underline{S}\,\underline{\dot{h}}_{t+\Delta t} = \underline{q}_{t+\Delta t} \qquad (2)$$

Linear head change in time Δt, gives

$$\underline{\dot{h}}_\tau = \frac{1}{\Delta t}\left(\underline{h}_{t+\Delta t} - \underline{h}_t\right) \qquad (3)$$

and $$\underline{\dot{h}}_{t+\Delta t} \simeq \frac{1}{\Delta t}\left(\underline{h}_{t+\Delta t} - \underline{h}_t\right) \qquad (4)$$

Substituting (4) into (2) and rearranging gives

$$\underline{K}\,\underline{h}_{t+\Delta t} + \underline{S}\frac{1}{\Delta t}\left(\underline{h}_{t+\Delta t} - \underline{h}_t\right) = \underline{q}_{t+\Delta t} \qquad (5)$$

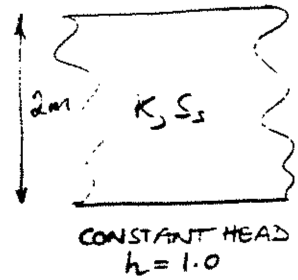
or. $$\left[\underline{K} + \frac{1}{\Delta t}\underline{S}\right]\underline{h}_{t+\Delta t} = \underline{q}_{t+\Delta t} + \frac{1}{\Delta t}\underline{S}\,\underline{h}_t \qquad (6)$$

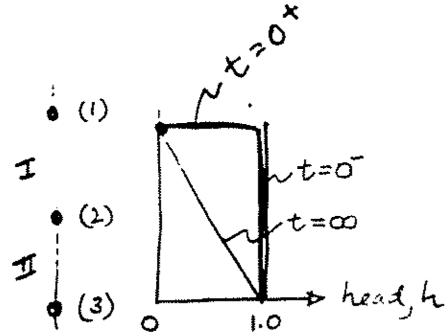
or $$\underline{K}^*\underline{h}_{t+\Delta t} = \underline{q}^*_{t+\Delta t}$$

Unconditionally stable method. $\quad (\lambda = 1.0)$

Example 2.4.4.1 One-Dimensional Problem

One dimensional flow in a body, initially at $h = 1.0$. At time $t = 0^+$ the topsurface head is changed to $h = 0$ and held constant. The basal head is retained at $h = 1.0$.





$$K = 1.0 \text{ m/s} \quad ; \quad S_s = \cancel{0.5}\ 2.0 \text{ m}^{-1} \quad ; \quad \Delta t = 0.1 \text{ s}$$

Using two node elements

$$\underline{K}_I = \underline{K}_{II} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1}$$

$$\underline{S}_I = \underline{S}_{II} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2}$$

The system matrices are

$$\underline{K}\,\underline{h}_\tau + \underline{S}\,\underline{\dot{h}}_\tau = \underline{q}_{r\tau} \tag{3}$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{\underline{K}} \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix}_\tau + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\underline{S}} \begin{Bmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{Bmatrix}_\tau = \begin{Bmatrix} q_1 \\ q_{r2} \\ q_{r3} \end{Bmatrix}_\tau \tag{4}$$

Since the heads at nodes 1 and 3 are known at time $t = 0^+$; $h_1 = 0$; $h_3 = 1.0$

Rearranging equation (4) gives

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ h_2 \\ 0 \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{Bmatrix}_{t+\Delta t} = \begin{Bmatrix} q_1 - h_1 \\ \cancel{q_2}^{\,0} + h_1 + h_3 \\ q_3 - h_3 \end{Bmatrix}_{t+\Delta t} \tag{5}$$

where only the second equation remains active. Rearranging to the form of equation (2.6.4.1.8) the system matries may be determined, or the actual active equations since h_1 and h_3 are known for all time.

Correspondingly

$$\underline{K}^* = [\underline{K} + \frac{1}{\Delta t}\underline{S}] = 2 + \frac{1}{0.1}(2) = 22 \tag{6}$$

$$q^*_{t+\Delta t} = \overset{\nearrow 0}{\cancel{q_2}} + h_1 + h_3 + \frac{1}{\Delta t} S h_2 \qquad (7)$$

$$= 0 + 0 + 1.0 + \frac{1}{0.1}(2) h_2 \qquad (h_2 = 1.0)$$

$$q^*_{t=0.1} = 21.0 \qquad (8)$$

The single equation is, therefore, from equations (6) and (8)

$$\underline{K}^* \underline{h}_{t+\Delta t} = \underline{q}^*_{t+\Delta t} \qquad (9)$$

$$22\, h_2 = 21 \qquad h_2 = 0.9545 \text{ @ } t+\Delta t = 0.1$$

For the next time step, $\underline{K}^*$, is the same and only $\underline{q}^*_{t+\Delta t}$ changes. For time level $t+\Delta t = 0.2$

$$q^*_{t+\Delta t} = \overset{\nearrow 0}{\cancel{q_{t+\Delta t}}} + \overset{\nearrow 0}{\cancel{h_1}} + \overset{\nearrow 1.0}{\cancel{h_3}} + \frac{1}{\Delta t} S \overset{\nearrow (21/22)}{\cancel{h_t}} \qquad (10)$$

$$q^*_{t+\Delta t} = 0 + 0 + 1.0 + \frac{1}{0.1}(2)\left(\frac{21}{22}\right) =$$

$$\underline{K}^* \underline{h}_{t+\Delta t} = \underline{q}^*_{t+\Delta t} \qquad h_2 = 0.9132 \text{ @ } t+\Delta t = 0.2$$

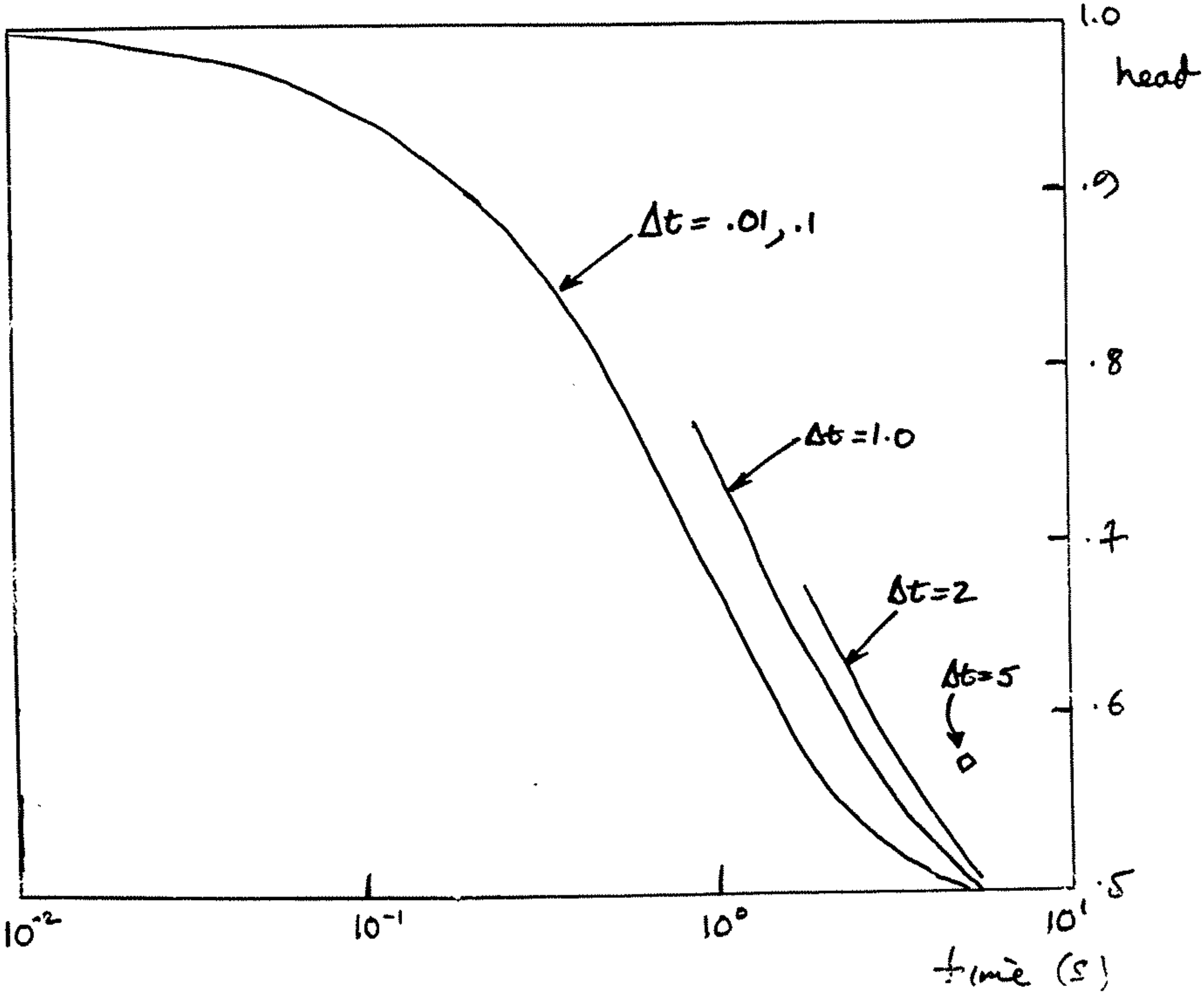
The recurrence relationship is therefore

$$h_{t+\Delta t} = \frac{1}{K^*}\left(h_3 + \frac{1}{\Delta t} S\, h_t\right) = \frac{1}{22}\left(1 + 20 h_t\right)$$

enabling the history of heads to be evaluated as illustrated graphically for time steps varying between $0.01 < \Delta t < 5.0$ in Figure 26.4.1.1

Although unconditionally stable, the solution converges as Δt approaches 0.1 s. Thus, time discretization is important as an aspect separate from stability. The results illustrated in the figure are only approximate. The will be as adequate as the spatial discretization.

IMPLICIT SOLUTION - Unconditionally stable



Solution converges for $\Delta t \leq .1$

$\therefore$ solution stable in time.

Not, however, exact solution since spatial discretization is coarse.

3:4 Transient Behavior - Integration in Time [2:7]

https://youtu.be/no5Y6dCXcyE

[2:7] Fluid Flow and Pressure Diffusion

Recap

Transient Behavior $\underline{K}\underline{h} + \underline{S}\dot{\underline{h}} = \underline{q}$

Time integration

EGEEfem

EXPLICIT TIME INTEGRATION

Instead, write equation at time $\tau = t$, then

$$\underline{K}\,\underline{h}_t + \underline{S}\,\dot{\underline{h}}_t = \underline{q}_t \qquad (1)$$

Form $\underline{S}$ as a lumped matrix (terms on diagonal only) then

$$\underline{S}^{-1} = \frac{1}{S} \qquad (2)$$

and (1) may be rearranged as

$$\dot{\underline{h}}_t = \underline{S}^{-1}[\underline{q}_t - \underline{K}\,\underline{h}_t] \qquad (3)$$

and the time derivative of head may also be evaluated as

$$\underline{h}_{t+\Delta t} = \underline{h}_t + \Delta t\,\dot{\underline{h}}_t \qquad (4)$$

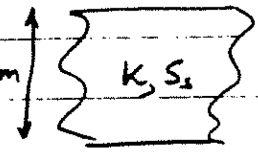
The magnitudes of heads may be evaluated as follows:

1. Evaluate $\underline{K}$ and $\underline{S}$
2. Evaluate $\dot{\underline{h}}_t$ from (3)
3. Evaluate $\underline{h}_{t+\Delta t}$ from (4)
4. Reevaluate $\dot{\underline{h}}_t$ with new heads where $\tau = t + \Delta t$

Conditionally stable.

Example 2.4.4.2 One-Dimensional Problem

Using the same physical parameters as Example 2.4.4.1.



Assembling the system in rearranged form $\underline{K}\,\underline{h}_t + \underline{S}\,\dot{\underline{h}}_t = \underline{q}_t$

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ h_2 \\ 0 \end{Bmatrix}_t + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{Bmatrix} = \begin{Bmatrix} q_1 - h_1 \\ q_2 + h_1 + h_3 \\ q_3 \quad - h_3 \end{Bmatrix} \qquad (1)$$

The single active equation is $\quad K h_2 + \underline{S}\dot{h}_2 = q_2 + h_1 + h_3 \qquad (2)$

Substituting into equation (2.4.4.2.3) gives

$$\dot{\underline{h}}_t = \underline{S}^{-1}[\,\underline{q}_{t+\Delta t} + h_1 + h_3 - \underline{K}\,\underline{h}_t\,] \qquad (3)$$

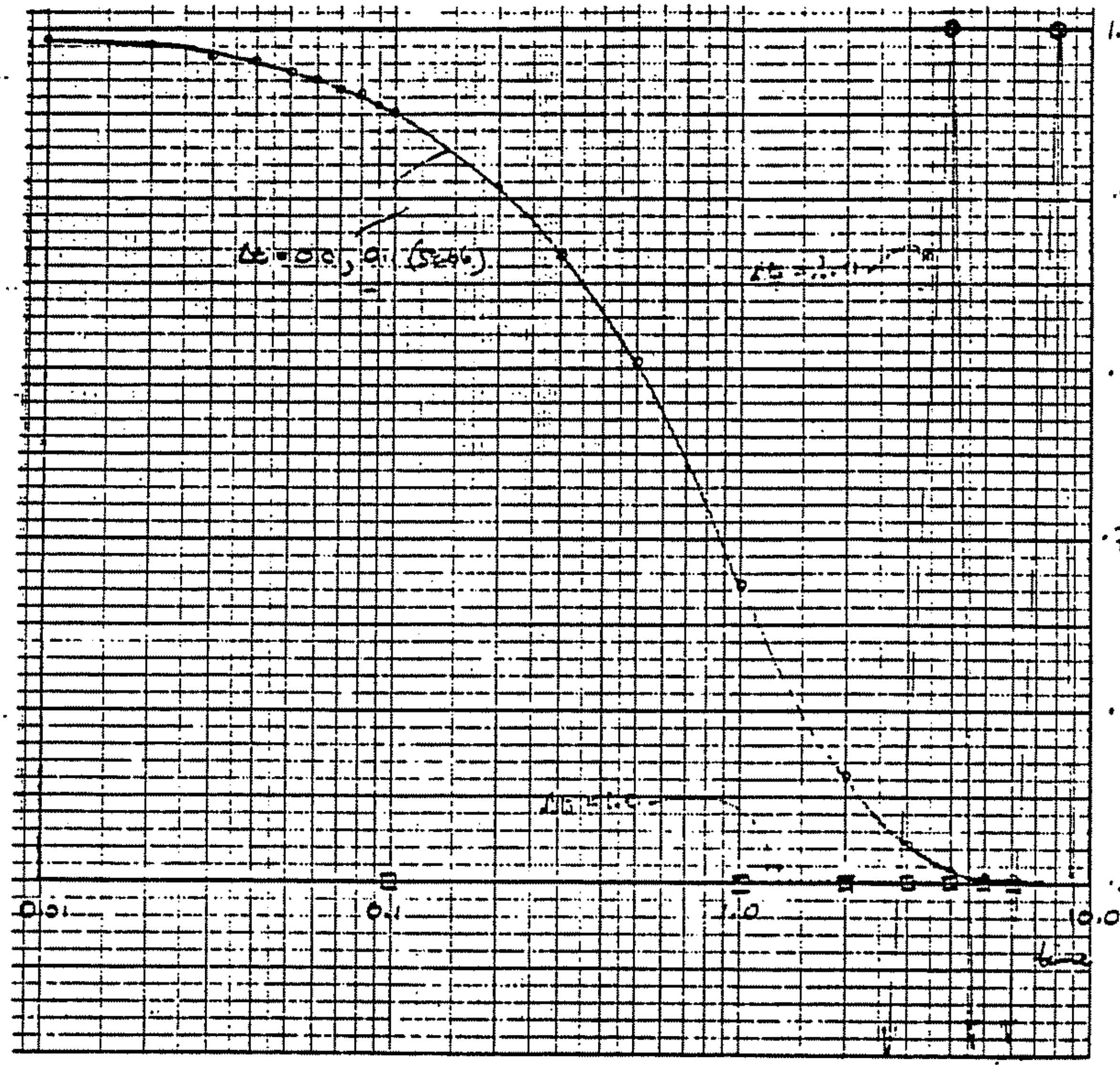
$$\dot{\underline{h}}_t = \tfrac{1}{2}[\,0 + 0 + \overset{1.0}{\cancel{R}} - (2)\underline{h}_t\,] \qquad (4)$$

$$\underline{h}_{t+\Delta t} = \underline{h}_t + \Delta t\,\dot{\underline{h}}_t \qquad (5)$$

For $\Delta t = 0.1s$ and other time step magnitudes the transient behavior of head at node 2 is given in Table 2.6.4.2.1 and shown graphically in Figure 2.4.4.2.1

The solution method is only conditionally stable. For $\Delta t \geq 1.0$ the solution oscillates.

The primary advantage is that a conductance matrix never has to be inverted or solved. This method is particularly suited to nonlinear problems where conductivity magnitudes change with time or head gradient.



Note – unstable in this instance for Δt > 1.0
Δt = 1.0 → steady, incorrect solution for h
Δt = 2.0 → oscillating solution

NOTE: Very good correspondence with impiecal solution

Figure 2.4.4.2.1 Transient response at node 2.

GENERAL TIME INTEGRATION

Define $$h_\tau = h_t + \lambda(h_{t+\Delta t} - h_t) \qquad (1)$$

Linear gradient as $$\underline{h}'_\tau = \frac{1}{\Delta t}(\underline{h}_{t+\Delta t} - \underline{h}_t) \qquad (2)$$

Substitute (1) and (2) into the following,

$$\underline{K}\,\underline{h}_\tau + \underline{S}\,\underline{h}'_\tau = \underline{q}_\tau \qquad (3)$$

To yield:

$$\underline{K}[(1-\lambda)\underline{h}_t + \lambda\underline{h}_{t+\Delta t}] + \frac{1}{\Delta t}\underline{S}(\underline{h}_{t+\Delta t} - \underline{h}_t) = \underline{q}_\tau \qquad (4)$$

This yields:

Implicit for	$\lambda = 1$	backward difference
Crank-Nicolson	$\lambda = \frac{1}{2}$	central difference
Explicit for	$\lambda = 0$	forward difference

$\lambda \geq \frac{1}{2}$ are unconditionally stable

$\lambda < \frac{1}{2}$ are conditionally stable.

3:5 Dual-Porosity-Dual-Permeability Models [6:1]

Lecture

https://youtu.be/key0TuOSDBU

[6:1] Linked Mechanisms

Dual Porosity/Dual Permeability

- Concept
- Dual permeability
  - Heuristic derivation
  - Comsol implementation

DUAL PERMEABILITY MODELS

FRACTURES AND POROUS BLOCKS HAVE DIFFERENT HYDRAULIC PARAMETERS AND THEREFORE RESPONSE TIMES.

- FRACTURES (high K, low S)
- MATRIX (low K, high S)

Double diffusion equations:

leakage parameter

specific surface area

$$K_1 \left[\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right] = S_1 \frac{\partial h_1}{\partial t} + \pi a^2 (h_1 - h_2) \qquad (1)$$

$$K_2 \left[\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} \right] = S_2 \frac{\partial h_2}{\partial t} - \pi a^2 (h_1 - h_2) \qquad (2)$$

h_1 = porous medium ; h_2 = fracture.

FEM equations:

$$\underline{K}_1 \underline{h}_1^{\tau} + \underline{S}_1 \dot{\underline{h}}_1^{\tau} + \underline{B} (\underline{h}_1^{\tau} - \underline{h}_2^{\tau}) = \underline{q}_1^{\tau}$$

$$\underline{K}_2 \underline{h}_2^{\tau} + \underline{S}_2 \dot{\underline{h}}_2^{\tau} - \underline{B} (\underline{h}_1^{\tau} - \underline{h}_2^{\tau}) = \underline{q}_2^{\tau}$$

$$\underline{B} = \pi a^2 \int b^T b \, dV$$

Add time integration; Implicit:

$$\lambda = 1.0$$

$$\tau = t + \Delta t$$

$$\dot{\underline{h}}_1^{\tau} = \frac{1}{\Delta t} \left(\underline{h}_1^{t+\Delta t} - \underline{h}_1^{t} \right) \text{ etc.}$$

$$\underline{K}_1 \underline{h}_1^{t+\Delta t} + \frac{1}{\Delta t} \underline{S}_1 (\underline{h}_1^{t+\Delta t} - \underline{h}_1^{t}) + \underline{B}(\underline{h}_1^{t+\Delta t} - \underline{h}_2^{t+\Delta t}) = \underline{q}_1^{t+\Delta t}$$

$$\underline{K}_2 \underline{h}_2^{t+\Delta t} + \frac{1}{\Delta t} \underline{S}_2 (\underline{h}_2^{t+\Delta t} - \underline{h}_2^{t}) - \underline{B}(\underline{h}_1^{t+\Delta t} - h_2^{t+\Delta t}) = \underline{q}_2^{t+\Delta t}$$

<u>Rearrange in matrix form as</u>

$$\begin{bmatrix} \underline{A}_1 & -\underline{A}_3 \\ -\underline{A}_3 & \underline{A}_2 \end{bmatrix} \begin{Bmatrix} \underline{h}_1 \\ \underline{h}_2 \end{Bmatrix}^{t+\Delta t} = \begin{Bmatrix} \underline{q}_1^{t+\Delta t} + \frac{1}{\Delta t} \underline{S}_1 \underline{h}_1^{t} \\ \underline{q}_2^{t+\Delta t} + \frac{1}{\Delta t} \underline{S}_2 \underline{h}_2^{t} \end{Bmatrix}$$

$$\underline{A}_1 = \underline{K}_1 + \frac{1}{\Delta t} \underline{S}_1 + \underline{B}$$

$$\underline{A}_2 = \underline{K}_2 + \frac{1}{\Delta t} \underline{S}_2 + \underline{B}$$

$$\underline{A}_3 = \underline{B}$$

$$\underline{K}_1 = \int_V \underline{a}^T \underline{D} \underline{a} \, dV$$

$$\underline{S}_1 = S_1 \int_V \underline{b}^T \underline{b} \, dV$$

$$\underline{B} = \pi a^2 \int_V \underline{b}^T \underline{b} \, dV$$

POROMECHANICS OF POROUS AND FRACTURED RESERVOIRS
Jishan Liu, Derek Elsworth
KIGAM, Daejeon, Korea
May 15-18, 2017

1. Poromechanics – Flow Properties (Jishan Liu)

1:1 Reservoir Pressure System – How to calculate overburden stress and reservoir pressure ***Day 1***[3]
1:2 Darcy's Law – Permeability and its changes, reservoir classification
1:3 Mass Conservation Law – flow equations
1:4 Steady-State Behaviors – Solutions of simple flow problems
1:5 Hydraulic Diffusivity – Definition, physical meaning, and its application in reservoirs
1:6 Rock Properties – Their Dependence on Stress Conditions ***Day 1***

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2. Poromechanics – Fluid Storage Properties (Jishan Liu)
2:1 Fluid Properties – How they change and affect flow ***Day 2***
2:2 Mechanisms of Liquid Production or Injection
2:3 Estimation of Original Hydrocarbons in Place
2:4 Estimation of Ultimate Recovery or Injection of Hydrocarbons
2:5 Flow – Deformation Coupling in Coal
2:6 Flow – Deformation Coupling in Shale ***Day 2***
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3. Poromechanics –Modeling Porous Medium Flows (Derek Elsworth)
3:1 Single porosity flows - Finite Element Methods [2:1] Lecture ***Day 3***
3:2 2D Triangular Constant Gradient Elements [2:3] Lecture
3:3 Transient Behavior - Mass Matrices [2:6] Lecture
3:4 Transient Behavior - Integration in Time [2:7] Lecture
3:5 Dual-Porosity-Dual-Permeability Models [6:1] Lecture ***Day 3***

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4. Poromechanics –Modeling Coupled Porous Medium Flow and Deformation (Derek Elsworth)
4:1 Mechanical properties – http://www.ems.psu.edu/~elsworth/courses/geoee500/GeoEE500_1.PDF ***Day 4***
4:2 Biot consolidation – http://www.ems.psu.edu/~elsworth/courses/geoee500/GeoEE500_1.PDF
4:3 Dual-porosity poroelasticity
4:4 Mechanical deformation - 1D and 2D Elements [5:1][5:2] Lecture
4:5 Coupled Hydro-Mechanical Models [6:2] Lecture ***Day 4***

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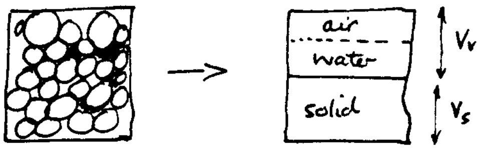
3 *All sessions nominally1h:15m*

4:1 Mechanical properties

http://www.ems.psu.edu/~elsworth/courses/geoee500/GeoEE500_1.PDF

DEFINITIONS



Unit weight of water $\qquad \gamma_w \qquad$ 1 gram/cc $\qquad 9.81\ KN/m^3$

Specific gravity $\qquad G_s \qquad$ typically 2.7

Unit wt. of grains $\qquad G_s \gamma_w$

Porosity $\qquad n = \dfrac{V_v}{V_v + V_s} \qquad e = \dfrac{n}{1-n}$

Void ratio $\qquad e = \dfrac{V_v}{V_s} \qquad n = \dfrac{e}{1+e}$

Degree of saturation $\qquad S = \dfrac{V_{water}}{V_v}$

Water content $\qquad w = \dfrac{\text{Wt of water}}{\text{Wt of solids}} = \dfrac{V_{water}\ \gamma_w}{V_{solid}\ G_s \gamma_w}$

$$= \frac{V_w}{V_v} \frac{V_v}{V_s} \frac{\cancel{\gamma_w}}{G \cancel{\gamma_w}} = \frac{Se}{G}$$

Unit wt. of soil $\qquad \gamma = \dfrac{\text{wt. of soil}}{\text{volume}} = \cancel{V_w G \gamma_w}$

R.F. CRAIG. "SOIL MECHANICS"

CHAPTER 7

Consolidation Theory

7.1 Introduction

As explained in Chapter 3, consolidation is the gradual reduction in volume of a fully saturated soil of low permeability due to drainage of some of the pore water, the process continuing until the excess pore water pressure set up by an increase in total stress has completely dissipated: the simplest case is that of one-dimensional consolidation, in which a condition of zero lateral strain is implicit. The process of swelling, the reverse of consolidation, is the gradual increase in volume of a soil under negative excess pore water pressure.

Consolidation settlement is the vertical displacement of the surface corresponding to the volume change at any stage of the consolidation process. Consolidation settlement will result, for example, if a structure is built over a layer of saturated clay or if the water table is lowered permanently in a stratum overlying a clay layer. If, on the other hand, an excavation is made in a saturated clay, heaving (the reverse of settlement) will result in the bottom of the excavation due to swelling of the clay. In cases in which significant lateral strain takes place, there will be an immediate settlement due to deformation of the soil under undrained conditions, in addition to consolidation settlement. Immediate settlement can be estimated using the results from elastic theory given in Chapter 5. This chapter is concerned with the prediction of both the magnitude and rate of consolidation settlement.

The progress of consolidation in situ can be monitored by installing piezometers to record the change in pore water pressure with time. The magnitude of settlement can be measured by recording the levels of suitable reference points on a structure or in the ground: precise levelling is essential, working from a benchmark which is not subject to even the slightest settlement. Every opportunity should be taken of obtaining settlement data as it is only through such measurements that the adequacy of theoretical methods can be assessed.

7.2 The Oedometer Test

The characteristics of a soil during one-dimensional consolidation or swelling can be determined by means of the oedometer test. Fig. 7.1 shows diagrammatically a cross-section through an oedometer. The test specimen is in the form of a disc, held inside a metal ring and lying between two porous stones. The upper porous stone, which can move inside the ring with a small clearance, is fixed below a metal loading cap through which pressure can be applied to the specimen. The whole assembly sits in an open cell of water to which the pore water in the specimen has free access. The ring confining the specimen may be either fixed (clamped to the body of the cell) or floating (being free to move vertically): the inside of the ring should have a smooth polished surface to reduce side friction. The confining ring imposes a condition of zero lateral strain on the specimen, the ratio of lateral to vertical effective stress being K_0, the coefficient of earth pressure at rest. The compression of the specimen under pressure is measured by means of a dial gauge operating on the loading cap.

The test procedure has been standardized in BS 1377 [7.4] which specifies that the oedometer shall be of the fixed ring type. The initial pressure will depend on the type of soil, then a sequence of pressures is applied to the specimen, each being double the previous value. Each pressure is normally maintained for a period of 24 hours (in exceptional cases a period of 48 hours may be required), compression readings being

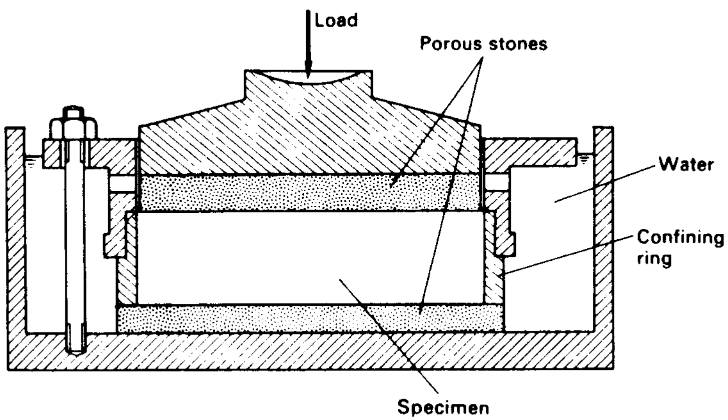


Fig. 7.1 The oedometer.

observed at suitable intervals during this period. At the end of the increment period, when the excess pore water pressure has completely dissipated, the applied pressure equals the effective vertical stress in the specimen. The results are presented by plotting the thickness (or percentage change in thickness) of the specimen or the void ratio at the end of each increment period against the corresponding effective stress. The effective stress may be plotted to either a natural or a logarithmic scale. If desired, the expansion of the specimen can be measured under successive decreases in applied pressure. However, even if the swelling characteristics of the soil are not required, the expansion of the specimen due to the removal of the final pressure should be measured.

The void ratio at the end of each increment period can be calculated from the dial gauge readings and either the water content or dry weight of the specimen at the end of the test. Referring to the phase diagram in Fig. 7.2, the two methods of calculation are as follows.

(1) Water content measured at end of test = w_1
Void ratio at end of test = $e_1 = w_1 G_s$ (assuming $S_r = 100\%$)
Thickness of specimen at start of test = H_0
Change in thickness during test = ΔH
Void ratio at start of test = $e_0 = e_1 + \Delta e$
where:

$$\frac{\Delta e}{\Delta H} = \frac{1 + e_0}{H_0} \tag{7.1}$$

In the same way Δe can be calculated up to the end of any increment period.

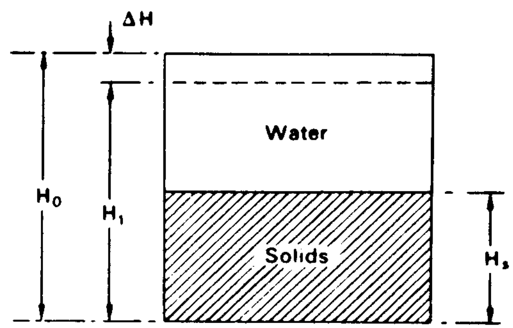


Fig. 7.2 Phase diagram.

(2) Dry weight measured at end of test = M_s (i.e. mass of solids)
Thickness at end of any increment period = H_1
Area of specimen = A
Equivalent thickness of solids = $H_s = M_s/AG_s\rho_w$
Void ratio,

$$e_1 = \frac{H_1 - H_s}{H_s} = \frac{H_1}{H_s} - 1 \tag{7.2}$$

Compressibility Characteristics

Typical plots of void ratio (e) after consolidation, against effective stress (σ') for a saturated clay are shown in Fig. 7.3, the plots showing an initial compression followed by expansion and recompression (cf. Fig. 4.10 for isotropic consolidation). The shapes of the curves are related to the stress history of the clay. The e-log σ' relationship for a normally consolidated clay is linear (or very nearly so) and is called the virgin compression line. If a clay is overconsolidated its state will be represented by a point on the expansion or recompression parts of the e-log σ' plot. The recompression curve ultimately joins the virgin compression line: further compression then occurs along the virgin line. During compression, changes in soil structure continuously take place and the clay does not revert to the original structure during expansion. The plots show that a clay in the overconsolidated state will be much less compressible than the same clay in a normally consolidated state.

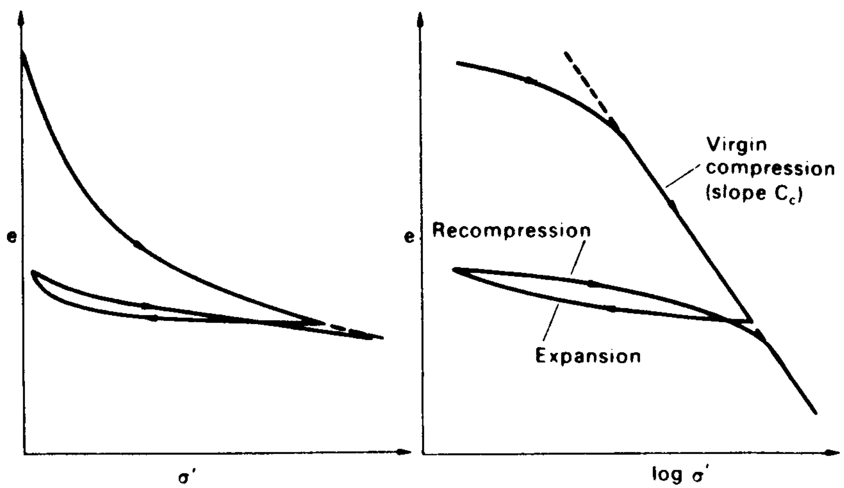


Fig. 7.3 Void ratio-effective stress relationship.

The compressibility of the clay can be represented by one of the following coefficients.

(*1*) *The coefficient of volume compressibility* (m_v), defined as the volume change per unit volume per unit increase in effective stress. The units of m_v are the inverse of pressure (m^2/MN). The volume change may be expressed in terms of either void ratio or specimen thickness. If, for an increase in effective stress from σ'_0 to σ'_1 the void ratio decreases from e_0 to e_1, then:

$$m_v = \frac{1}{1 + e_0}\left(\frac{e_0 - e_1}{\sigma'_1 - \sigma'_0}\right) \tag{7.3}$$

$$= \frac{1}{H_0}\left(\frac{H_0 - H_1}{\sigma'_1 - \sigma'_0}\right) \tag{7.4}$$

The value of m_v for a particular soil is not constant but depends on the stress range over which it is calculated. BS 1377 specifies the use of the m_v coefficient calculated for a stress increment of 100 kN/m^2 in excess of the effective overburden pressure of the in-situ soil at the depth of interest, although the coefficient may also be calculated, if required, for any other stress range.

(*2*) *The compression index* (C_c) is the slope of the linear portion of the e-log σ' plot and is dimensionless. For any two points on the linear portion of the plot:

$$C_c = \frac{e_0 - e_1}{\log \dfrac{\sigma'_1}{\sigma'_0}} \tag{7.5}$$

The expansion part of the e-log σ' plot can be approximated to a straight line the slope of which is referred to as the *expansion index* C_e.

Preconsolidation Pressure

Casagrande proposed an empirical construction to obtain from the e-log σ' curve for an overconsolidated clay the maximum effective vertical stress that has acted on the clay in the past, referred to as the *preconsolidation*

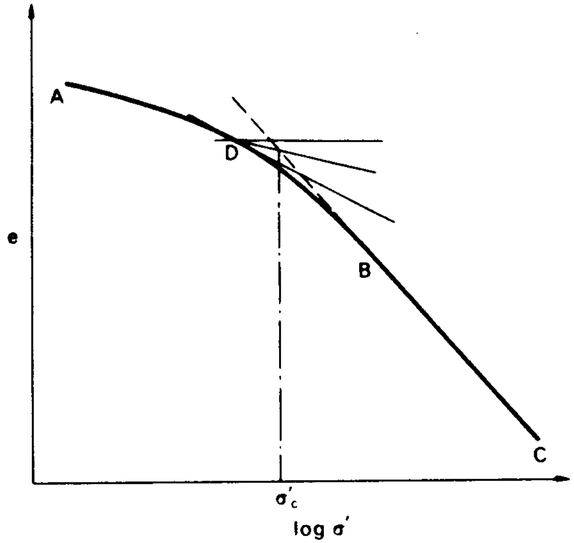


Fig. 7.4 Determination of preconsolidation pressure.

pressure (σ'_c). Fig. 7.4 shows a typical *e*-log σ' curve for a specimen of clay, initially overconsolidated. The initial curve indicates that the clay is undergoing recompression in the oedometer, having at some stage in its history undergone expansion. Expansion of the clay in situ may, for example, have been due to melting of ice sheets, erosion of overburden or a rise in water table level. The construction for estimating the preconsolidation pressure consists of the following steps.

1. Produce back the straight line part (BC) of the curve.
2. Determine the point (D) of maximum curvature on the recompression part (AB) of the curve.
3. Draw the tangent to the curve at D and bisect the angle between the tangent and the horizontal through D.
4. The vertical through the point of intersection of the bisector and CB produced gives the approximate value of the preconsolidation pressure.

Whenever possible the preconsolidation pressure for an overconsolidated clay should not be exceeded in construction. Compression will not usually be great if the effective vertical stress remains below σ'_c: only if σ'_c is exceeded will compression be large.

In-Situ e-log σ′ Curve

Due to the effects of sampling and preparation the specimen in an oedometer test will be slightly disturbed. It has been shown that an increase in the degree of specimen disturbance results in a slight decrease in the slope of the virgin compression line. It can therefore be expected that the slope of the line representing virgin compression of the in-situ soil will be slightly greater than the slope of the virgin line obtained in a laboratory test.

No appreciable error will be involved in taking the in-situ void ratio as being equal to the void ratio (e_0) at the start of the laboratory test. Schmertmann [7.17] pointed out that the laboratory virgin line may be expected to intersect the in-situ virgin line at a void ratio of approximately 0·42 times the initial void ratio. Thus the in-situ virgin line can be taken as the line EF in Fig. 7.5 where the coordinates of E are log σ'_c and e_0, and F is the point on the laboratory virgin line at a void ratio of 0·42 e_0.

In the case of overconsolidated clays the in-situ condition is represented by the point (G) having coordinates σ'_0 and e_0, where σ'_0 is the present effective overburden pressure. The in-situ recompression curve can be approximated to the straight line GH parallel to the mean slope of the laboratory recompression curve.

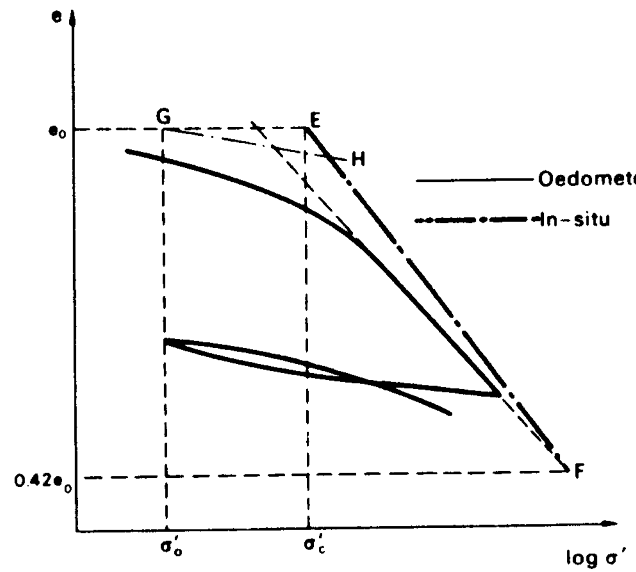
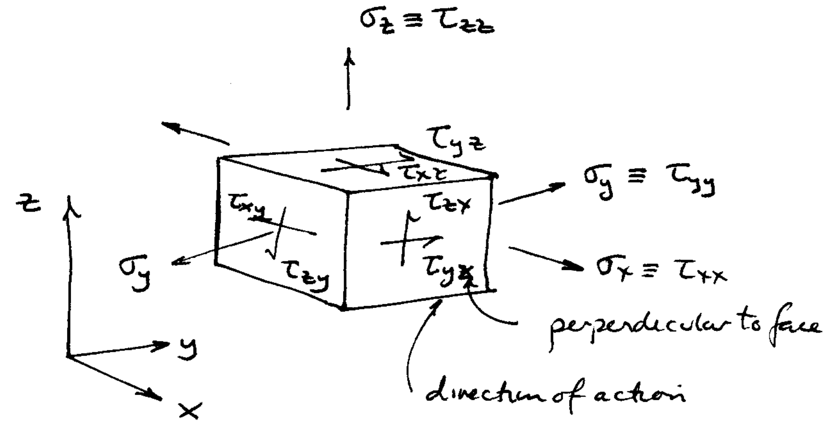


Fig. 7.5 In-situ *e*-log σ' curve.

EQUILIBRIUM EQUATIONS



Force balance yields:
$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho g_x + \frac{\partial^2 v_x}{\partial t^2}$$

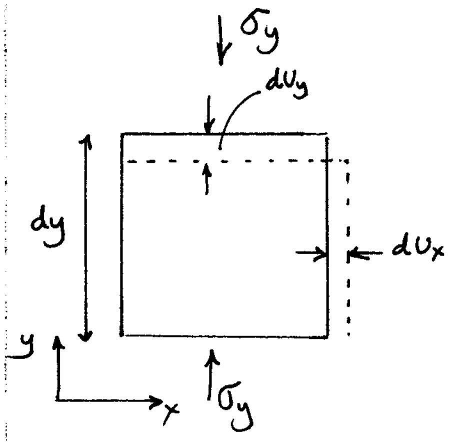
(3 equations)

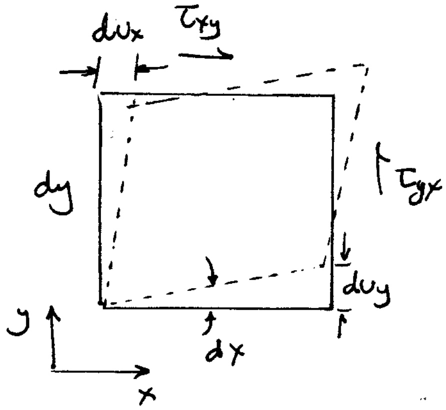
Consistent with conservation of momentum:

$$\underbrace{\frac{\partial}{\partial t}\rho v_x + \left(\frac{\partial}{\partial x}\rho v_x v_x + \frac{\partial}{\partial y}\rho v_y v_x + \frac{\partial}{\partial z}\rho v_z v_x\right)}_{\rho\, Dv_x/Dt}$$
$$= -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x$$

$$\cancel{\rho \frac{Dv_x}{Dt}}^{\,0} = -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \cancel{\frac{\partial p}{\partial x}}^{\,0} + \rho g_x$$

HOOKE'S LAW





$$\varepsilon_y = \frac{\partial U_y}{\partial y} \simeq \frac{\Delta U_y}{\Delta y}$$

$$\gamma_{xy} = \left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right)$$

$$\nu = -\varepsilon_x \frac{E}{\sigma_y}$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z) \right]$$
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z) \right]$$
$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_y + \sigma_x) \right]$$

$$\left.\begin{aligned} \gamma_{xy} &= \tau_{xy}/G \\ \gamma_{yz} &= \tau_{yz}/G \\ \gamma_{zx} &= \tau_{zx}/G \end{aligned}\right\} \quad G = \frac{E}{2(1+\nu)}$$

Create inverse relations: $\varepsilon = f(\sigma, p) \longrightarrow \sigma = f(\varepsilon, p)$

$\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$

Define volume strain:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \varepsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \varepsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned}\right\} \quad \begin{aligned} \varepsilon_v &= \frac{1}{E}[3\sigma_m - 2\nu(3\sigma_m)] \\ \varepsilon_v &= \frac{3}{E}[\sigma_m - 2\nu\sigma_m] \end{aligned}$$

$$\varepsilon_v = \frac{3\sigma_m}{E}[1 - 2\nu]$$

$$\sigma_m = \frac{\varepsilon_v E}{3(1-2\nu)}$$

Rewrite Hooke's Law:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_x = \frac{1}{E}[\sigma_x + \nu\sigma_x - \nu\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_x = \frac{1}{E}[(1+\nu)\sigma_x - \nu(\sigma_x + \sigma_y + \sigma_z)] = \frac{1}{E}[(1+\nu)\sigma_x - \nu(3\sigma_m)]$$

Rearrange in terms of σ_x:

Substitute for σ_m

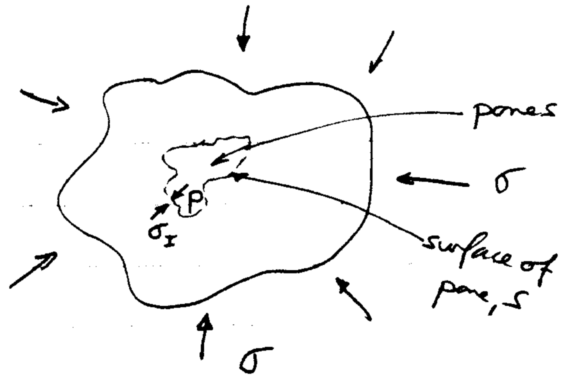
$$\frac{\varepsilon_x E}{(1+\nu)} + \frac{\nu 3\sigma_m}{(1+\nu)} = \sigma_x$$

$$G = \frac{E}{2(1+\nu)}$$

$$\frac{2}{2}\frac{\varepsilon_x E}{(1+\nu)} + \frac{2}{2}\frac{3\nu}{(1+\nu)}\frac{\varepsilon_v E}{3(1-2\nu)} = \sigma_x$$

$$2G\,\varepsilon_x + 2G\frac{\nu\,\varepsilon_v}{(1-2\nu)} = \sigma_x$$

Nur & Byerlee (1971)



Consider body of connected pores

Apply confining pressure, σ and uniform pore pressure, p $\Big\}$ $\sigma \neq p$

Apply load in two steps

1. Apply pore pressure, p and equal confining stress σ_I $\Big\}$ $\sigma_I = p$ (1)

2. Apply remaining confining stress, $\sigma_{II} = \sigma - p$ (2) without changing pore pressure

Calculate strain fields

A. Volumetric strain of the aggregate due to step 2. is

$$\frac{\Delta v}{v}_{II} = \beta \sigma_{II} = \beta(\sigma - p) \quad (3)$$

where $\beta = (1/K)$ is effective compressibility of the dry aggregate
drained

B. Volumetric strain due to step 1.
Since $\sigma_I = p$ are the applied stresses
Assume that the pores are filled with the solid and a confining stress σ_I applied. Any contour around the grain must also have stress $\sigma_I = p$ acting normal to that boundary.
The strain is therefore

$$\frac{\Delta v}{v_I} = \beta_s \sigma_I = \beta_s p \quad (4)$$

$\beta_s = \left(\frac{1}{K_s}\right)$. is the compressibility of the solid (no pores)

Remove the solid material; keep stress σ_I acting or p since $\sigma_I = p$

Invoke Uniqueness Theorem [e.g. Mushkelishvili, 1963]

" Deformation of a body is uniquely determined when normal stresses are applied on all boundaries."

Since $\sigma_I = p$ then replacing solid inclusions with liquid causes no deformation.

$$\frac{\Delta V}{V_I} = \beta_s \sigma_I = \beta_s p \; . \qquad (4)$$

Total strain $$\frac{\Delta V}{V} = \frac{\Delta V}{V_I} + \frac{\Delta V}{V_{II}} = \beta_s p + \beta(\sigma - p)$$

$$\frac{\Delta V}{V} = \beta\sigma - (\beta - \beta_s) p$$

In terms of bulk modulii,

$$\frac{\Delta V}{V} = \left(\frac{1}{K}\right)\sigma - \left(\frac{1}{K} - \frac{1}{K_s}\right) p$$

Multiply through by K

$$\underbrace{K\frac{\Delta V}{V}}_{\sigma'} = \sigma - \underbrace{\left(1 - \frac{K}{K_s}\right)}_{\alpha} p$$

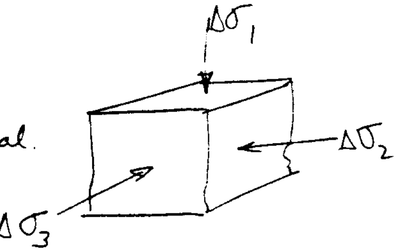
Same as Skempton postulated (1960)

* Experiment confirmed validity
* Laughton's experiment didn't pick up this point since $K \ll K_s$

PORE PRESSURE COEFFICIENTS (A & B)

Define the pore pressures that are developed as a result of undrained loading. $B = f(\text{hydrostatic } \sigma)$

$A = f(\text{deviatoric } \sigma)$



Assume Terzaghi effective stresses, α may be general.

$$\left.\begin{aligned} \Delta\sigma_1' &= \Delta\sigma_1 - \alpha\Delta p \\ \Delta\sigma_2' &= \Delta\sigma_2 - \alpha\Delta p \\ \Delta\sigma_3' &= \Delta\sigma_3 - \alpha\Delta p \end{aligned}\right\} \quad (1) \qquad \alpha = (1 - K/K_s)$$

Isotropic, homogeneous

Elastic medium, ε = strains ; E = modulus ; ν = Poisson Ratio

$$\left.\begin{aligned} \Delta\varepsilon_1 &= \frac{1}{E}[\Delta\sigma_1' - \nu(\Delta\sigma_2' + \Delta\sigma_3')] \\ \Delta\varepsilon_2 &= \frac{1}{E}[\Delta\sigma_2' - \nu(\Delta\sigma_1' + \Delta\sigma_3')] \\ \Delta\varepsilon_3 &= \frac{1}{E}[\Delta\sigma_3' - \nu(\Delta\sigma_1' + \Delta\sigma_2')] \end{aligned}\right\}$$

strain controlled by effective stress. (2)

Total volumetric strain

$$\Delta\varepsilon_1 + \Delta\varepsilon_2 + \Delta\varepsilon_3 = \frac{\Delta V}{V} = \frac{(1-2\nu)}{E}[\Delta\sigma_1' + \Delta\sigma_2' + \Delta\sigma_3'] \qquad (3)$$

Isotropic stresses $\qquad \Delta\sigma_1' = \Delta\sigma_2' = \Delta\sigma_3' = \Delta\sigma'$

then $$\frac{\Delta V}{V} = C_c\,\Delta\sigma' \qquad (4)$$ C_c = coef of compressibility

$$\frac{\Delta V}{V} = \underbrace{\frac{3(1-2\nu)}{E}}\,\Delta\sigma' \qquad (5)$$

From equation (3)

$$\frac{\Delta V}{V} = \frac{C_c}{3}[\Delta\sigma_1' + \Delta\sigma_2' + \Delta\sigma_3'] \qquad (6)$$

$$= \frac{C_c}{3}[\Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3 - 3\alpha\Delta p] \qquad (7)$$

But $\Delta V = n V C_f \Delta p \quad (8)$ $\qquad$ n = porosity

C_f = fluid compressibility

Substituting (8) into (7)

$$\frac{n V C_f \Delta p}{V} = \frac{C_c}{3}\left[\Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3 - 3\alpha\Delta p\right] \qquad (9)$$

$$\Delta p\left(n C_f + \alpha C_c\right) = \frac{C_c}{3}\left[\Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3\right]$$

$$\left(\alpha + n\frac{C_f}{C_c}\right)\Delta p = \frac{1}{3}\left[\Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3\right]$$

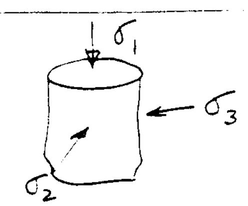
$$\Delta p = \frac{1}{\left(\alpha + n\frac{C_f}{C_c}\right)}\,\frac{1}{3}\left[\Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3\right] \qquad (10)$$

Soil mechanics $\alpha = 1$

$$\Delta p = B\,\frac{1}{3}\left[\Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3\right] \qquad (11)$$

B gives Δp due to undrained loading by 'mean stress' $\frac{1}{3}[\Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3]$

Consider undrained triaxial compression test



$\sigma_3 = \sigma_2$

From (11)

$$\Delta p = B\,\frac{1}{3}\left[\Delta\sigma_1 + \Delta\sigma_2 + \Delta\sigma_3\right] = B\,\frac{1}{3}\left[\Delta\sigma_1 + 2\Delta\sigma_3\right]$$

$$\Delta p = B\,\frac{1}{3}\left[\Delta\sigma_1 - \Delta\sigma_3 + 3\Delta\sigma_3\right]$$

$$\Delta p = B\,\Delta\sigma_3 + B\,\frac{1}{3}\left[\Delta\sigma_1 - \Delta\sigma_3\right]$$

$B \simeq 1.0$ for $S = 100\%$ ie $1/(1 + \frac{nC_f}{C_c})$

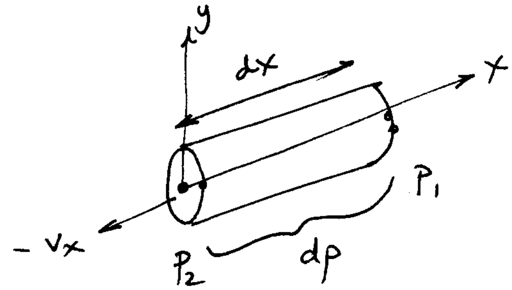
$$\Delta p = \underbrace{B\,\Delta\sigma_3}_{\text{hydrostatic}} + \underbrace{B.A\left[\Delta\sigma_1 - \Delta\sigma_3\right]}_{\text{deviatoric (failure)}}$$

$$A\begin{cases}\text{NC} & 1.0\\ \text{OC}\times 4 & .3 - .2\\ \text{OC}\times 12 & .2 - 0\end{cases}$$

Since soil not elastic, $A \neq \frac{1}{3}$, but evaluate structural parameter from tests.

DARCY'S LAW

$$v_x = -k \frac{\partial p}{\partial x}$$



Conservation of Mass:

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot \rho \underline{v}$$

Eulerian:
$$\frac{\partial \rho}{\partial t} = -\left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) - \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

$$v_x = -k \frac{\partial p}{\partial x}$$

Lagrangian:
$$\frac{D\rho}{Dt} = -\rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

$$\frac{D\rho}{Dt} = +\rho k \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right)$$

R.F. CRAIG – "SOIL MECHANICS"

7.7 Terzaghi's Theory of One-Dimensional Consolidation

The assumptions made in the theory are:

1. The soil is homogeneous.
2. The soil is fully saturated.
3. The solid particles and water are incompressible.
4. Compression and flow are one-dimensional (vertical).
5. Strains are small.
6. Darcy's law is valid at all hydraulic gradients.
7. The coefficient of permeability and the coefficient of volume compressibility remain constant throughout the process.
8. There is a unique relationship, independent of time, between void ratio and effective stress.

Regarding assumption 6, there is evidence of deviation from Darcy's law at low hydraulic gradients. Regarding assumption 7, the coefficient of permeability decreases as the void ratio decreases during consolidation. The coefficient of volume compressibility also decreases during consolidation since the e-σ' relationship is non-linear. However for small stress increments assumption 7 is reasonable. The main limitations of Terzaghi's theory (apart from its one-dimensional nature) arise from assumption 8. Experimental results show that the relationship between void ratio and effective stress is not independent of time.

The theory relates the following three quantities.

1. The *excess* pore water pressure (u).
2. The depth (z) below the top of the clay layer.
3. The time (t) from the instantaneous application of a total stress increment.

Consider an element having dimensions dx, dy and dz within a clay layer of thickness $2d$, as shown in Fig. 7.16. An increment of total vertical stress $\Delta\sigma$ is applied to the element.

The flow velocity through the element is given by Darcy's law as

$$v_z = ki_z = -k\frac{\partial h}{\partial z}$$

Since any change in total head (h) is due only to a change in pore water pressure:

$$v_z = -\frac{k}{\gamma_w}\frac{\partial u}{\partial z}$$

The condition of continuity (Equation 2.7) can therefore be expressed as

$$-\frac{k}{\gamma_w}\frac{\partial^2 u}{\partial z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\mathrm{d}V}{\mathrm{d}t} \tag{7.14}$$

The rate of volume change can be expressed in terms of m_v:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = m_v\frac{\partial \sigma'}{\partial t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

The total stress increment is gradually transferred to the soil skeleton,

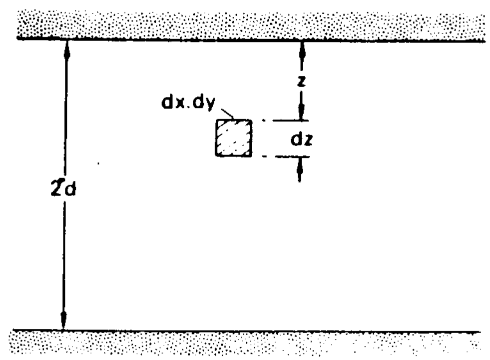


Fig. 7.16 Element within a clay layer.

increasing effective stress, as the excess pore water pressure decreases. Hence the rate of volume change can be expressed as

$$\frac{dV}{dt} = -m_v \frac{\partial u}{\partial t} dx dy dz \tag{7.15}$$

Combining Equations 7.14 and 7.15,

$$m_v \frac{\partial u}{\partial t} = \frac{k}{\gamma_w} \frac{\partial^2 u}{\partial z^2}$$

or

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} \tag{7.16}$$

This is the differential equation of consolidation, in which

$$c_v = \frac{k}{m_v \gamma_w} \tag{7.17}$$

c_v being defined as the *coefficient of consolidation*, suitable units being m^2/year. Since k and m_v are assumed constant, c_v is constant during consolidation.

Solution of the Consolidation Equation

The total stress increment is assumed to be applied instantaneously, and at zero time will be carried entirely by the pore water, i.e. the initial value of excess pore water pressure (u_i) is equal to $\Delta\sigma$ and the initial condition is:

$$u = u_i \quad \text{for } 0 \leqslant z \leqslant 2d \quad \text{when } t = 0$$

The upper and lower boundaries of the clay layer are assumed to be free draining, the permeability of the soil adjacent to each boundary being very high compared to that of the clay. Thus the boundary conditions at any time after the application of $\Delta\sigma$ are:

$$u = 0 \quad \text{for } z = 0 \quad \text{and } z = 2d \quad \text{when } t > 0$$

The solution for the excess pore water pressure at depth z after time t is:

$$u = \sum_{n=1}^{n=\infty} \left(\frac{1}{d} \int_0^{2d} u_i \sin \frac{n\pi z}{2d} dz \right) \left(\sin \frac{n\pi z}{2d} \right) \exp\left(-\frac{n^2 \pi^2 c_v t}{4d^2} \right) \tag{7.18}$$

where d = length of longest drainage path, and u_i = initial excess pore water pressure, in general a function of z.

For the particular case in which u_i is constant throughout the clay layer:

$$u = \sum_{n=1}^{n=\infty} \frac{2u_i}{n\pi} (1 - \cos n\pi) \left(\sin \frac{n\pi z}{2d} \right) \exp\left(-\frac{n^2 \pi^2 c_v t}{4d^2} \right) \tag{7.19}$$

When n is even, $(1 - \cos n\pi) = 0$, and when n is odd, $(1 - \cos n\pi) = 2$. Only odd values of n are therefore relevant and it is convenient to make the substitutions:

$$n = 2m + 1$$

and

$$M = \frac{\pi}{2}(2m + 1)$$

It is also convenient to substitute

$$T_v = \frac{c_v t}{d^2} \tag{7.20}$$

a dimensionless number called the *time factor*. Equation 7.19 then becomes

$$u = \sum_{m=0}^{m=\infty} \frac{2u_i}{M} \left(\sin \frac{Mz}{d} \right) \exp(-M^2 T_v) \tag{7.21}$$

The progress of consolidation can be shown by plotting a series of curves of u against z for different values of t. Such curves are called *isochrones* and their form will depend on the initial distribution of excess pore water pressure and the drainage conditions at the boundaries of the clay layer. A layer for which both the upper and lower boundaries are free-draining is described as an *open* layer: a layer for which only one boundary is free-draining is a *half-closed* layer. Examples of isochrones are shown in Fig. 7.17. In part (a) of the figure the initial distribution of u_i is constant and for an open layer of thickness $2d$ the isochrones are symmetrical about the centre line. The upper half of this diagram also represents the case of a half-closed layer of thickness d. The slope of an isochrone at any depth gives the hydraulic gradient and also indicates the direction of flow. In parts (b) and (c) of the figure, with a triangular distribution of u_i, the direction of flow changes over certain parts of the layer. In part (c) the lower boundary is impermeable and for a time swelling takes place in the lower part of the layer.

The degree of consolidation at depth z and time t can be obtained by substituting the value of u (Equation 7.21) in Equation 7.13, giving

$$U_z = 1 - \sum_{m=0}^{m=\infty} \frac{2}{M} \left(\sin \frac{Mz}{d} \right) \exp(-M^2 T_v) \tag{7.22}$$

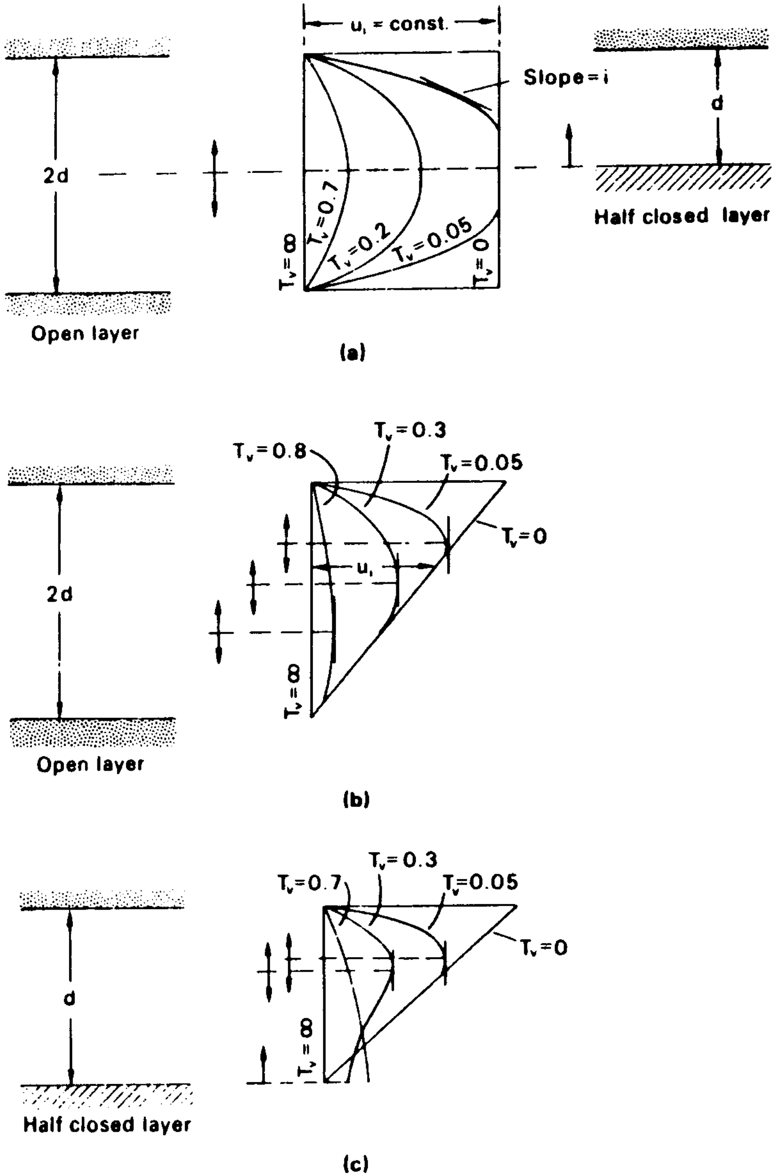


Fig. 7.17 Isochrones.

In practical problems it is the *average* degree of consolidation (U) over the depth of the layer as a whole that is of interest, the consolidation settlement at time t being given by the product of U and the final settlement. The average degree of consolidation at time t for constant u_i is given by

$$U = 1 - \frac{\dfrac{1}{2d}\displaystyle\int_0^{2d} u\,\mathrm{d}z}{u_i}$$

$$= 1 - \sum_{m=0}^{m=\infty} \frac{2}{M^2} \exp(-M^2 T_v) \tag{7.23}$$

The relationship between U and T_v given by Equation 7.23 is represented by curve 1 in Fig. 7.18. Equation 7.23 can be represented almost exactly by the following empirical equations:

$$\text{for } U < 0{\cdot}60, \quad T_v = \frac{\pi}{4} U^2 \tag{7.24a}$$

$$\text{for } U > 0{\cdot}60, \quad T_v = -0{\cdot}933 \log(1 - U) - 0{\cdot}085 \tag{7.24b}$$

If u_i is not constant the average degree of consolidation is given by

$$U = 1 - \frac{\displaystyle\int_0^{2d} u\,\mathrm{d}z}{\displaystyle\int_0^{2d} u_i\,\mathrm{d}z} \tag{7.25}$$

where

$$\int_0^{2d} u\,\mathrm{d}z = \text{area under isochrone at the time in question}$$

and

$$\int_0^{2d} u_i\,\mathrm{d}z = \text{area under initial isochrone}$$

(For a half-closed layer the limits of integration are 0 and d in the above equations.)

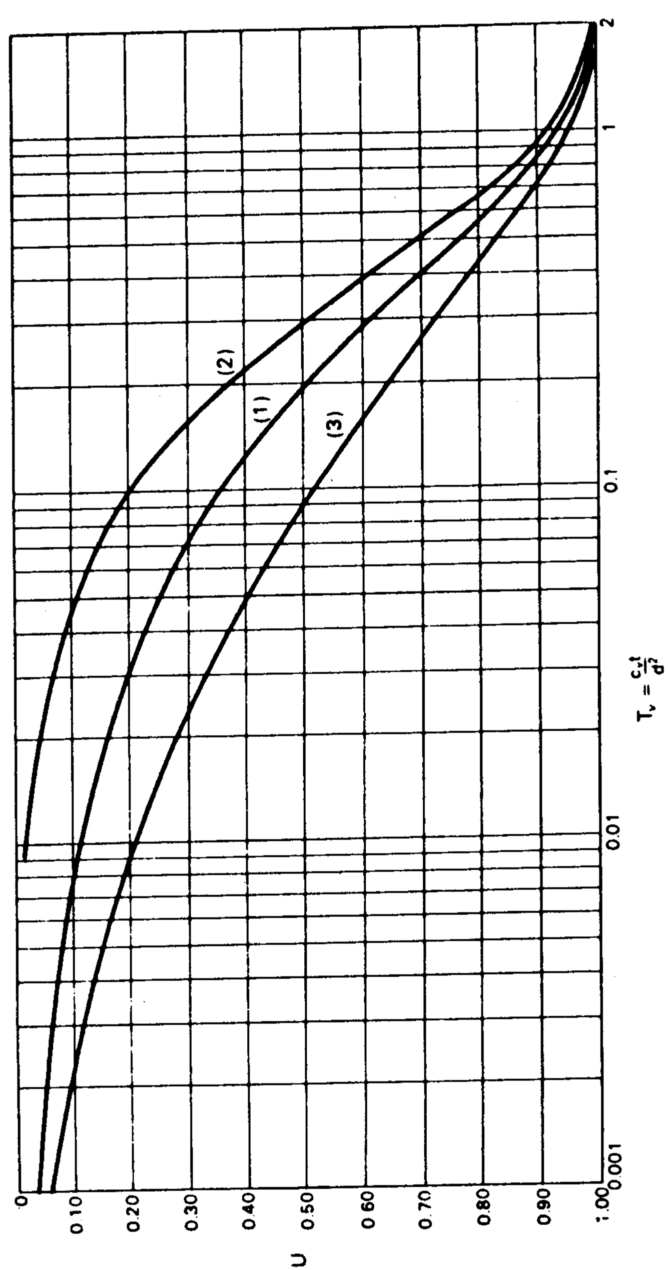


Fig. 7.18 Relationships between average degree of consolidation and time factor.

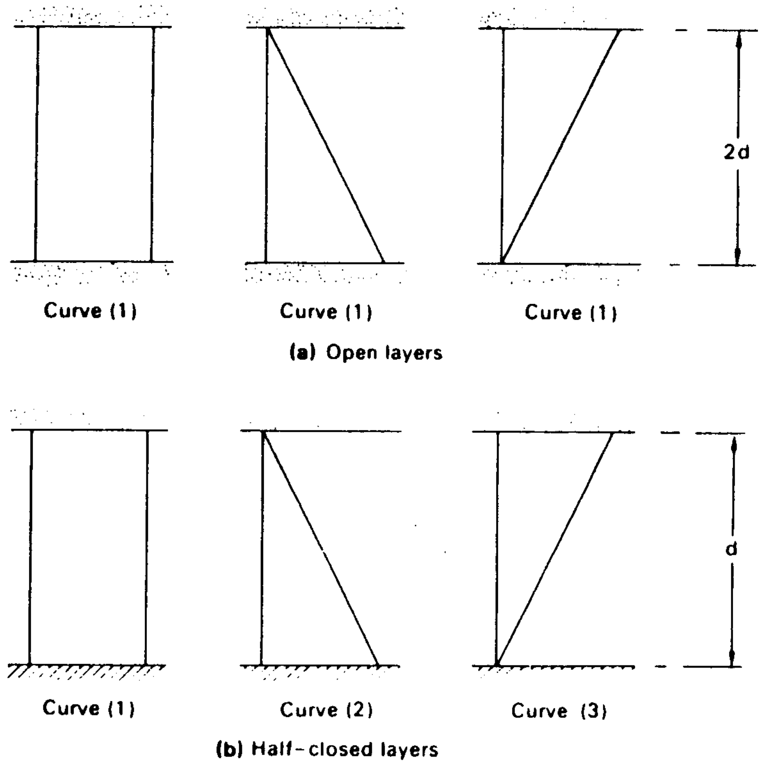


Fig. 7.19 Initial variations of excess pore water pressure.

The initial variation of excess pore water pressure in a clay layer can usually be approximated in practice to a linear distribution. Curves 1, 2 and 3 in Fig. 7.18 represent the solution of the consolidation equation for the cases shown in Fig. 7.19.

7.8 Determination of Coefficient of Consolidation

The value of c_v for a particular pressure increment in the oedometer test can be determined by comparing the characteristics of the experimental and theoretical consolidation curves, the procedure being referred to as *curve fitting*. The characteristics of the curves are brought out clearly if time is plotted to a square root or a logarithmic scale. Once the value of c_v has been determined, the coefficient of permeability can be calculated from Equation 7.17, the oedometer test being a useful method for obtaining the permeability of a clay.

The Log Time Method (due to Casagrande)

The forms of the experimental and theoretical curves are shown in Fig. 7.20. The experimental curve is obtained by plotting the dial gauge readings in the oedometer test against the logarithm of time in minutes. The theoretical curve is given as the plot of the average degree of consolidation against the logarithm of the time factor. The theoretical curve consists of three parts: an initial curve which approximates closely to a parabolic relationship, a part which is linear and a final curve to which the horizontal axis is an asymptote at $U = 1{\cdot}0$ (or 100%). In the experimental curve the point corresponding to $U = 0$ can be determined by using the fact that the initial part of the curve represents an approximately parabolic relationship between compression and time. Two points on the curve are selected (A and B in Fig. 7.20) for which the values of t are in the ratio of 4:1, and the vertical distance between them is measured. An equal distance set off above the first point fixes the point (a_s) corresponding to $U = 0$. As a check the procedure should be repeated using different pairs of points. The point corresponding to $U = 0$ will not generally correspond to the point (a_0) representing the initial dial gauge reading, the difference being due mainly to the compression of small quantities of air in the soil, the degree of saturation being marginally below 100%: this compression is called *initial compression*. The final part of the experimental curve is linear but not horizontal and the point (a_{100}) corresponding to $U = 100\%$ is taken as the intersection of the two linear parts of the curve. The compression between the a_s and a_{100} points is called *primary consolidation* and represents that part of the process accounted for by Terzaghi's theory. Beyond the point of intersection, compression of the soil continues at a very slow rate for an indefinite period of time and is called *secondary compression.*

The point corresponding to $U = 50\%$ can be located midway between the a_s and a_{100} points and the corresponding time t_{50} obtained. The value of T_v corresponding to $U = 50\%$ is 0·196 and the coefficient of consolidation is given by

$$c_v = \frac{0{\cdot}196\, d^2}{t_{50}} \tag{7.26}$$

the value of d being taken as half the average thickness of the specimen for the particular pressure increment. BS 1377 states that if the average temperature of the soil in situ is known and differs from the average test temperature, a correction should be applied to the value of c_v, correction factors being given in the standard.

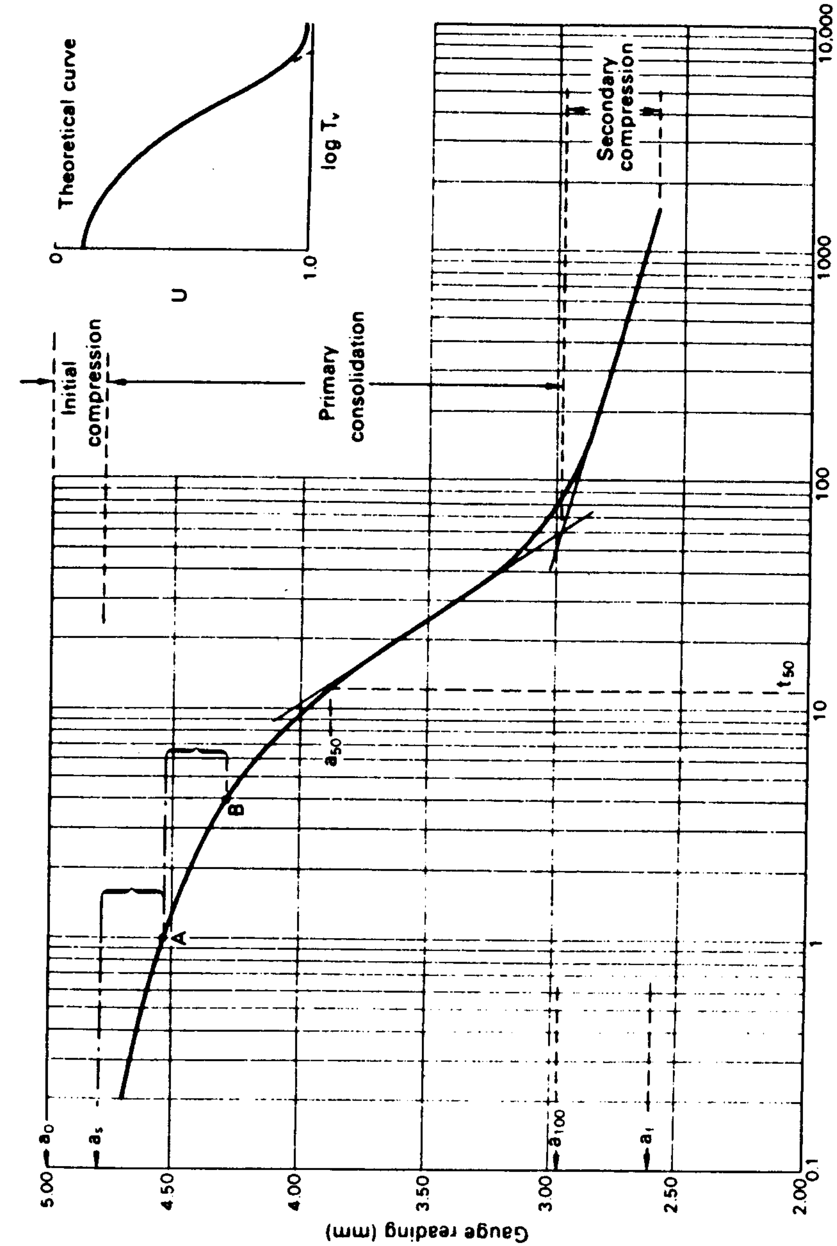


Fig. 7.20 The log time method.

4:2 Biot consolidation

http://www.ems.psu.edu/~elsworth/courses/geoee500/GeoEE500_1.PDF

Pergamon

Int. J. Solids Structures Vol. 35, Nos 34-35, pp. 4515-4517, 1998

Printed in Great Britain
0020-7683/98/$—see front matter

A TRIBUTE TO MAURICE A. BIOT

Maurice A. Biot (1905–1985)

In presenting Maurice Anthony Biot with the Timoshenko Medal in 1962, R. D. Mindlin, the eminent Professor at Columbia University, wrote: "Fundamentally, Tony Biot has a strong consciousness of the physical world around him. He has a keen insight which enables him to recognize the essential features of a physical phenomenon and build them into a mathematical model without blindly including non-essentials. Then he has, at his fingertips, a vast array of the tools of mathematical analysis and analytical methods of approximation which he uses skillfully to extract, from the model, predictions of the hitherto unpredictable. They are all too few such men these days." These words by Mindlin accurately portrayed M. A. Biot as an intuitive engineer, who could master the advanced tools of a physical scientist, and as a scientist who did not lose sight of the physical world.

Maurice Biot was born in Antwerp, Belgium on May 25th 1905. The war in 1914–1918 and the siege of Antwerp caused the Biot family to travel first to London, then Paris, and finally settling in Chambéry, France. These moves matured the young Biot and exposed him to several languages.

Later returning to Antwerp, M. Biot concluded his secondary school. In 1923 he enrolled at a school in Brussels for preparatory courses in mathematics, and in 1924 was admitted at the Université catholique de Louvain. It was at this time that Biot showed his insatiable appetitie for knowledge. While pursuing his studies in Engineering, Biot also attended courses in Philosophy (he was awarded a Bachelor degree in Philosophy in 1927) and Economics. He obtained a Mining Engineering degree in 1929, and an Electrical Engineering degree in 1930.

After defending his thesis entitled "Theoretical studies on induced electrical currents", Biot was awarded a Doctor of Science degree in 1931. The sponsorship of the Belgian American Educational Foundation allowed Biot to spend the next two years in the U.S. (1931–1933) at the California Institute of Technology in Pasadena. It was at Cal Tech where he first met and worked with Theodore von Kármán, who had arrived in the U.S. in 1929. Biot acquired a Ph.D. in Aeronautical Sciences in 1932 by defending his work "Concept of response spectrum based on the earthquake acceleration." The methodology brought great simplifications to the analysis of structures under transient loading and has since been used as a tool in earthquake-proof design. It was during the same period that he published his first papers on a new approach to the nonlinear theory of elasticity accounting for the effect of initial stress. By that time he had published about a dozen scientific papers and patented his first three inventions.

After a few months at the University of Michigan, Biot returned to Europe. In 1933 and 1934, the Belgian National Scientific Research Foundation granted him the opportunity to travel to Delft, Göttingen, and Zurich. With such sharp intelligence he was soon recognized by the university community. In 1934, Biot started his academic career as a teacher of applied mathematics at Harvard University. In June 1935, he returned to Pasadena as an Advanced Fellow of the Belgian American Educational Foundation. By 1936, Biot was elected to the faculty at his Alma Mater, the Université catholique de Louvain, where he taught Elasticity and Analytical Mechanics. From 1937 to 1946, Biot was a Professor of Theoretical Mechanics and Physical Mathematics at Columbia University. In 1946 Brown University offered him the position of Professor in Applied Physics and Sciences, which he held until 1952.

It was in 1940 that the monograph *Mathematical Methods in Engineering* was written with Th. von Kármán. Its translation into nine languages is evidence of its influence on several generations of engineers. Later in his career he wrote two more books: *Mechanics of Incremental Deformations* (1965) and *Variational Principles in Heat Transfer* (1970).

The U.S. fascinated Biot, who found in it an environment conducive for research. Biot became an American citizen in 1941. The war in Europe came to Biot as a major distress and he took an active role in it. On leave from Columbia University, he worked at the Cal Tech Aeronautical Laboratory on problems of vibration and flutter, on the dynamic stability of projectiles and also on anti-submarine shell impact. During the war, as a Lieutenant Commander in the U.S. Navy, Biot headed the Structural Dynamics Section of the Bureau of Aeronautics in Washington, D.C. (1943–1945), and later conducted technical missions in Europe with combat troops.

By 1951, Biot had produced a large number of scientific works for Shell Development Co., Cornell Aeronautical Laboratory, and for the U.S. Air Force. After 1952 Biot worked largely alone as a consultant for various governmental agencies and industrial laboratories. From 1969 to 1982, Biot was a consultant for Mobil Research and Development Corporation in Dallas, in the area of Rock Mechanics.

Relocated in Brussels since 1970, Maurice Biot continued his research until his last day. It was on one of his last trips to the U.S. that Biot felt the early signs of his illness that would suddenly deprive him of his life on the 12th of September 1985, at the age of eighty.

The work and original contribution which distinguished Biot's career cover an unusually broad range of science and technology including applied mechanics, sound, heat, thermodynamics, aeronautics, geophysics, and electromagnetism. The level of his work has ranged from the highly theoretical and mathematical to practical applications and patented inventions.

Aeronautical problems and fluid mechanics were the objects of most of his efforts during the 1940s. He developed the three-dimensional aerodynamics theory of oscillating airfoils along with new methods of vibration analysis based on matrix theory and generalized coordinates. This led to widely applied design procedures of complex aircraft structures in order to prevent catastrophic flutter. He also patented an electrical analogue flutter predictor based on a simple circuit design which simulates aerodynamic forces. After the war he continued work on non-stationary aerodynamic instability of thin supersonic wings, and on the first evaluation of the transonic drag of an accelerated body.

In the 1950s, Biot's work was concerned primarily with problems in solid mechanics, porous media, thermodynamics, and heat transfer. He developed a new approach to the thermodynamics of irreversible processes by introducing a generalized form of the free energy as a key potential. The formulation was associated with new variational principles and Lagrangian-type equations. The results with the introduction of internal coordinates provided the thermodynamics foundation of a general theory of anisotropic viscoelasticity and thermoelasticity. He later gave a systematic presentation of this work in a monograph *Variational Principles in Heat Transfer* in 1970 and indicated its applicability to many other problems such as those of elastic aquifers or neutron diffusion in nuclear reactor design.

Biot's interest in the mechanics of porous media dated back to 1940 with a fundamental paper in soil mechanics and consolidation. He returned to the subject in the 1950s in the more general context of rock mechanics in connection with problems in the oil industry.

On the basis of his earlier work in thermodynamics, he extended his theory to the acoustics in porous media and showed that there existed three types of acoustic waves in such media. In another contribution he was the first person to correctly provide the solution of the so-called Stoneley wave, i.e. an interface wave between a fluid overlaying an elastic solid half-space which, as some have argued, should more appropriately be named after him.

For a short period in the middle 1950s Biot became involved with rocket radio-guidance problems and the question of disturbance from ground reflections. He showed that the reflection of electromagnetic and acoustic waves from a rough surface may be replaced by a smooth boundary condition. He also introduced a new approach to pulse generated transient waves based on a continuous spectrum of normal coordinates. The combination of the two methods provided the only practical solution at that time of some important problems.

In a series of papers starting in 1957, Biot extended his earlier work in the mechanics of initially stressed solids, developing a mathematical theory of folding instability of stratified viscous and viscoelastic solids. He verified the results in the laboratory and applied them to explain the dominant features of geological structures. In particular, he brought to light the phenomenon of internal buckling of a confined anisotropic or stratified medium under compressive stress and provided a quantitative analysis. He applied the theory with the same success to problems of gravity instability and salt dome formation. In a later period he presented a systematic treatment of the mechanics of initially stressed continua in the monograph *Mechanics of Incremental Deformations*, published in 1965. In the 1970s Biot's formulation of the variational principle of virtual dissipation in the thermodynamics of irreversible processes along with a new approach to open systems led to a synthesis of classical mechanics and irreversible thermodynamics. He applied these new theories to directly obtain the field equations in systems where deformations are coupled to thermo-molecular diffusion and chemical reactions. On this basis he also further developed the theory of porous media including heat and mass transport with phase changes and adsorption effects.

The honors that Biot received during his lifetime included the Timoshenko Medal of the American Society of Mechanical Engineers (1962), the Th. von Kármán Medal of the American Society of Civil Engineers (1967), and an Honorary Fellow of the Acoustical Society of America (1983). He was also a member of the U.S. National Academy of Engineering.

(Compilation based on the "Note on Maurice Anthony Biot" by A. Delmer and A. Jaumotte published in 1990 by the Académie Royale de Belgique, and on material supplied by Madame M. A. Biot.)

A. H.-D. Cheng
E. Detournay
Y. Abousleiman

General Theory of Three-Dimensional Consolidation*

MAURICE A. BIOT
Columbia University, New York, New York

(Received October 25, 1940)

The settlement of soils under load is caused by a phenomenon called consolidation, whose mechanism is known to be in many cases identical with the process of squeezing water out of an elastic porous medium. The mathematical physical consequences of this viewpoint are established in the present paper. The number of physical constants necessary to determine the properties of the soil is derived along with the general equations for the prediction of settlements and stresses in three-dimensional problems. Simple applications are treated as examples. The operational calculus is shown to be a powerful method of solution of consolidation problems.

INTRODUCTION

It is well known to engineering practice that a soil under load does not assume an instantaneous deflection under that load, but settles gradually at a variable rate. Such settlement is very apparent in clays and sands saturated with water. The settlement is caused by a gradual adaptation of the soil to the load variation. This process is known as *soil consolidation*. A simple mechanism to explain this phenomenon was first proposed by K. Terzaghi.[1] He assumes that the grains or particles constituting the soil are more or less bound together by certain molecular forces and constitute a porous material with elastic properties. The voids of the elastic skeleton are filled with water. A good example of such a model is a rubber sponge saturated with water. A load applied to this system will produce a gradual settlement, depending on the rate at which the water is being squeezed out of the voids. Terzaghi applied these concepts to the analysis of the settlement of a column of soil under a constant load and prevented from lateral expansion. The remarkable success of this theory in predicting the settlement for many types of soils has been one of the strongest incentives in the creation of a science of soil mechanics.

Terzaghi's treatment, however, is restricted to the one-dimensional problem of a column under a constant load. From the viewpoint of mathematical physics two generalizations of this are possible: the extension to the three-dimensional case, and the establishment of equations valid for any arbitrary load variable with time. The theory was first presented by the author in rather abstract form in a previous publication.[2] The present paper gives a more rigorous and complete treatment of the theory which leads to results more general than those obtained in the previous paper.

The following basic properties of the soil are assumed: (1) isotropy of the material, (2) reversibility of stress-strain relations under final equilibrium conditions, (3) linearity of stress-strain relations, (4) small strains, (5) the water contained in the pores is incompressible, (6) the water may contain air bubbles, (7) the water flows through the porous skeleton according to Darcy's law.

Of these basic assumptions (2) and (3) are most subject to criticism. However, we should keep in mind that they also constitute the basis of Terzaghi's theory, which has been found quite satisfactory for the practical requirements of engineering. In fact it can be imagined that the grains composing the soil are held together in a certain pattern by surface tension forces and tend to assume a configuration of minimum potential energy. This would especially be true for the colloidal particles constituting clay. It seems reasonable to assume that for small strains, when the grain pattern is not too much disturbed, the assumption of reversibility will be applicable.

The assumption of isotropy is not essential and

* Publication assisted by the Ernest Kempton Adams Fund for Physical Research of Columbia University.

[1] K. Terzaghi, *Erdbaumechanik auf Bodenphysikalischer Grundlage* (Leipzig F. Deuticke, 1925); "Principle of soil mechanics," Eng. News Record (1925), a series of articles.

[2] M. A. Biot, "Le problème de la Consolidation des Matières argileuses sous une charge," Ann. Soc. Sci. Bruxelles B55, 110–113 (1935).

anisotropy can easily be introduced as a refinement. Another refinement which might be of practical importance is the influence, upon the stress distribution and the settlement, of the state of initial stress in the soil before application of the load. It was shown by the present author[3] that this influence is greater for materials of low elastic modulus. Both refinements will be left out of the present theory in order to avoid undue heaviness of presentation.

The first and second sections deal mainly with the mathematical formulation of the physical properties of the soil and the number of constants necessary to describe these properties. The number of these constants including Darcy's permeability coefficient is found equal to five in the most general case. Section 3 gives a discussion of the physical interpretation of these various constants. In Sections 4 and 5 are established the fundamental equations for the consolidation and an application is made to the one-dimensional problem corresponding to a standard soil test. Section 6 gives the simplified theory for the case most important in practice of a soil completely saturated with water. The equations for this case coincide with those of the previous publication.[2] In the last section is shown how the mathematical tool known as the *operational calculus* can be applied most conveniently for the calculation of the settlement without having to calculate any stress or water pressure distribution inside the soil. This method of attack constitutes a major simplification and proves to be of high value in the solution of the more complex two- and three-dimensional problems. In the present paper applications are restricted to one-dimensional examples. A series of applications to practical cases of two-dimensional consolidation will be the object of subsequent papers.

1. Soil Stresses

Consider a small cubic element of the consolidating soil, its sides being parallel with the coordinate axes. This element is taken to be large enough compared to the size of the pores so that it may be treated as homogeneous, and at the same time small enough, compared to the scale of the macroscopic phenomena in which we are interested, so that it may be considered as infinitesimal in the mathematical treatment.

The average stress condition in the soil is then represented by forces distributed uniformly on the faces of this cubic element. The corresponding stress components are denoted by

$$\begin{matrix} \sigma_x & \tau_z & \tau_y \\ \tau_z & \sigma_y & \tau_x \\ \tau_y & \tau_x & \sigma_z. \end{matrix} \tag{1.1}$$

They must satisfy the well-known equilibrium conditions of a stress field.

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_z}{\partial y}+\frac{\partial \tau_y}{\partial z}=0, \\ &\frac{\partial \tau_z}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_x}{\partial z}=0, \\ &\frac{\partial \tau_y}{\partial x}+\frac{\partial \tau_x}{\partial y}+\frac{\partial \sigma_z}{\partial z}=0. \end{aligned} \tag{1.2}$$

Physically we may think of these stresses as composed of two parts; one which is caused by the hydrostatic pressure of the water filling the pores, the other caused by the average stress in the skeleton. In this sense the stresses in the soil are said to be carried partly by the water and partly by the solid constituent.

2. Strain Related to Stress and Water Pressure

We now call our attention to the strain in the soil. Denoting by u, v, w the components of the displacement of the soil and assuming the strain to be small, the values of the strain components are

$$\begin{aligned} e_x&=\frac{\partial u}{\partial x}, & \gamma_x&=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}, \\ e_y&=\frac{\partial v}{\partial y}, & \gamma_y&=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}, \\ e_z&=\frac{\partial w}{\partial z}, & \gamma_z&=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}. \end{aligned} \tag{2.1}$$

In order to describe completely the macroscopic condition of the soil we must consider an addi-

[3] M. A. Biot, "Nonlinear theory of elasticity and the linearized case for a body under initial stress."

tional variable giving the amount of water in the pores. We therefore denote by θ the increment of water volume per unit volume of soil and call this quantity the *variation in water content*. The *increment of water pressure* will be denoted by σ.

Let us consider a cubic element of soil. The water pressure in the pores may be considered as uniform throughout, provided either the size of the element is small enough or, if this is not the case, provided the changes occur at sufficiently slow rate to render the pressure differences negligible.

It is clear that if we assume the changes in the soil to occur by reversible processes the macroscopic condition of the soil must be a definite function of the stresses and the water pressure i.e., the seven variables

$$e_x \quad e_y \quad e_z \qquad \gamma_x \quad \gamma_y \quad \gamma_z \quad \theta$$

must be definite functions of the variables:

$$\sigma_x \quad \sigma_y \quad \sigma_z \qquad \tau_x \quad \tau_y \quad \tau_z \quad \sigma.$$

Furthermore if we assume the strains and the variations in water content to be small quantities, the relation between these two sets of variables may be taken as linear in first approximation. We first consider these functional relations for the particular case where $\sigma=0$. The six components of strain are then functions only of the six stress components $\sigma_x\ \sigma_y\ \sigma_z\ \tau_x\ \tau_y\ \tau_z$. Assuming the soil to have isotropic properties these relations must reduce to the well-known expressions of Hooke's law for an isotropic elastic body in the theory of elasticity; we have

$$\begin{aligned}
e_x&=\frac{\sigma_x}{E}-\frac{\nu}{E}(\sigma_y+\sigma_z),\\
e_y&=\frac{\sigma_y}{E}-\frac{\nu}{E}(\sigma_z+\sigma_x),\\
e_z&=\frac{\sigma_z}{E}-\frac{\nu}{E}(\sigma_x+\sigma_y),\\
\gamma_x&=\tau_x/G,\\
\gamma_y&=\tau_y/G,\\
\gamma_z&=\tau_z/G.
\end{aligned}\tag{2.2}$$

In these relations the constants E, G, ν may be interpreted, respectively, as Young's modulus, the shear modulus and Poisson's ratio for the solid skeleton. There are only two distinct constants because of the relation

$$G=\frac{E}{2(1+\nu)}.\tag{2.3}$$

Suppose now that the effect of the water pressure σ is introduced. First it cannot produce any shearing strain by reason of the assumed isotropy of the soil; second for the same reason its effect must be the same on all three components of strain $e_x\ e_y\ e_z$. Hence taking into account the influence of σ relations (2.2) become

$$\begin{aligned}
e_x&=\frac{\sigma_x}{E}-\frac{\nu}{E}(\sigma_y+\sigma_z)+\frac{\sigma}{3H},\\
e_y&=\frac{\sigma_y}{E}-\frac{\nu}{E}(\sigma_z+\sigma_x)+\frac{\sigma}{3H},\\
e_z&=\frac{\sigma_z}{E}-\frac{\nu}{E}(\sigma_x+\sigma_y)+\frac{\sigma}{3H},\\
\gamma_x&=\tau_x/G,\\
\gamma_y&=\tau_y/G,\\
\gamma_z&=\tau_z/G,
\end{aligned}\tag{2.4}$$

where H is an additional physical constant. These relations express the six strain components of the soil as a function of the stresses in the soil and the pressure of the water in the pores. We still have to consider the dependence of the increment of water content θ on these same variables. The most general relation is

$$\theta=a_1\sigma_x+a_2\sigma_y+a_3\sigma_z+a_4\tau_x+a_5\tau_y+a_6\tau_z+a_7\sigma.\tag{2.5}$$

Now because of the isotropy of the material a change in sign of $\tau_x\ \tau_y\ \tau_z$ cannot affect the water content, therefore $a_4=a_5=a_6=0$ and the effect of the shear stress components on θ vanishes. Furthermore all three directions x, y, z must have equivalent properties $a_1=a_2=a_3$. Therefore relation (2.5) may be written in the form

$$\theta=\frac{1}{3H_1}(\sigma_x+\sigma_y+\sigma_z)+\frac{\sigma}{R},\tag{2.6}$$

where H_1 and R are two physical constants.

Relations (2.4) and (2.6) contain five distinct physical constants. We are now going to prove that this number may be reduced to four; in fact that $H=H_1$ if we introduce the assumption of the existence of a potential energy of the soil. This assumption means that if the changes occur at an infinitely slow rate, the work done to bring the soil from the initial condition to its final state of strain and water content, is independent of the way by which the final state is reached and is a definite function of the six strain components and the water content. This assumption follows quite naturally from that of reversibility introduced above, since the absence of a potential energy would then imply that an indefinite amount of energy could be drawn out of the soil by loading and unloading along a closed cycle.

The potential energy of the soil per unit volume is

$$U=\tfrac{1}{2}(\sigma_x e_x+\sigma_y e_y+\sigma_z e_z+\tau_x\gamma_x+\tau_y\gamma_y+\tau_z\gamma_z+\sigma\theta). \quad (2.7)$$

In order to prove that $H=H_1$ let us consider a particular condition of stress such that

$$\sigma_x=\sigma_y=\sigma_z=\sigma_1,$$
$$\tau_x=\tau_y=\tau_z=0.$$

Then the potential energy becomes

$$U=\tfrac{1}{2}(\sigma_1\epsilon+\sigma\theta) \quad \text{with} \quad \epsilon=e_x+e_y+e_z$$

and Eqs. (2.4) and (2.6)

$$\epsilon=\frac{3(1-2\nu)}{E}\sigma_1+\frac{\sigma}{H}, \quad \theta=\sigma_1/H_1+\sigma/R. \quad (2.8)$$

The quantity ϵ represents the volume increase of the soil per unit initial volume. Solving for σ_1 and σ

$$\sigma_1=\frac{\epsilon}{R\Delta}-\frac{\theta}{H\Delta},$$

$$\sigma=\frac{-\epsilon}{H_1\Delta}+\frac{3(1-2\nu)\theta}{E\Delta}, \quad (2.9)$$

$$\Delta=\frac{3(1-2\nu)}{ER}-\frac{1}{HH_1}.$$

The potential energy in this case may be considered as a function of the two variables ϵ, θ. Now we must have

$$\frac{\partial U}{\partial \epsilon}=\sigma_1, \quad \frac{\partial U}{\partial \theta}=\sigma.$$

Hence

$$\frac{\partial \sigma_1}{\partial \theta}=\frac{\partial \sigma}{\partial \epsilon}$$

or

$$\frac{1}{H\Delta}=\frac{1}{H_1\Delta}.$$

We have thus proved that $H=H_1$ and we may write

$$\theta=\frac{1}{3H}(\sigma_x+\sigma_y+\sigma_z)+\frac{\sigma}{R}. \quad (2.10)$$

Relations (2.4) and (2.10) are the fundamental relations describing completely in first approximation the properties of the soil, for strain and water content, under equilibrium conditions. They contain four distinct physical constants G, ν, H and R. For further use it is convenient to express the stresses as functions of the strain and the water pressure σ. Solving Eq. (2.4) with respect to the stresses we find

$$\sigma_x=2G\left(e_x+\frac{\nu\epsilon}{1-2\nu}\right)-\alpha\sigma,$$

$$\sigma_y=2G\left(e_y+\frac{\nu\epsilon}{1-2\nu}\right)-\alpha\sigma,$$

$$\sigma_z=2G\left(e_z+\frac{\nu\epsilon}{1-2\nu}\right)-\alpha\sigma, \quad (2.11)$$

$$\tau_x=G\gamma_x,$$

$$\tau_y=G\gamma_y,$$

$$\tau_z=G\gamma_z$$

with

$$\alpha=\frac{2(1+\nu)}{3(1-2\nu)}\frac{G}{H}.$$

In the same way we may express the variation in water content as

$$\theta=\alpha\epsilon+\sigma/Q, \quad (2.12)$$

where

$$\frac{1}{Q}=\frac{1}{R}-\frac{\alpha}{H}.$$

3. Physical Interpretation of the Soil Constants

The constants E, G and ν have the same meaning as Young's modulus the shear modulus and the Poisson ratio in the theory of elasticity provided time has been allowed for the excess water to squeeze out. These quantities may be considered as the average elastic constants of the solid skeleton. There are only two distinct such constants since they must satisfy relation (2.3). Assume, for example, that a column of soil supports an axial load $p_0 = -\sigma_z$ while allowed to expand freely laterally. If the load has been applied long enough so that a final state of settlement is reached, i.e., all the excess water has been squeezed out and $\sigma = 0$ then the axial strain is, according to (2.4),

$$e_z = -\frac{p_0}{E} \tag{3.1}$$

and the lateral strain

$$e_x = e_y = \frac{\nu p_0}{E} = -\nu e_z. \tag{3.2}$$

The coefficient ν measures the ratio of the lateral bulging to the vertical strain under final equilibrium conditions.

To interpret the constants H and R consider a sample of soil enclosed in a thin rubber bag so that the stresses applied to the soil be zero. Let us drain the water from this soil through a thin tube passing through the walls of the bag. If a negative pressure $-\sigma$ is applied to the tube a certain amount of water will be sucked out. This amount is given by (2.10)

$$\theta = -\frac{\sigma}{R}. \tag{3.3}$$

The corresponding volume change of the soil is given by (2.4)

$$\epsilon = -\frac{\sigma}{H}. \tag{3.4}$$

The coefficient $1/H$ is a measure of the compressibility of the soil for a change in water pressure, while $1/R$ measures the change in water content for a given change in water pressure. The two elastic constants and the constants H and R are the four distinct constants which under our assumption define completely the physical proportions of an isotropic soil in the equilibrium conditions.

Other constants have been derived from these four. For instance α is a coefficient defined as

$$\alpha = \frac{2(1+\nu)}{3(1-2\nu)} \frac{G}{H}. \tag{3.5}$$

According to (2.12) it measures the ratio of the water volume squeezed out to the volume change of the soil if the latter is compressed while allowing the water to escape ($\sigma = 0$). The coefficient $1/Q$ defined as

$$\frac{1}{Q} = \frac{1}{R} - \frac{\alpha}{H} \tag{3.6}$$

is a measure of the amount of water which can be forced into the soil under pressure while the volume of the soil is kept constant. It is quite obvious that the constants α and Q will be of significance for a soil not completely saturated with water and containing air bubbles. In that case the constants α and Q can take values depending on the degree of saturation of the soil.

The standard soil test suggests the derivation of additional constants. A column of soil supports a load $p_0 = -\sigma_z$ and is confined laterally in a rigid sheath so that no lateral expansion can occur. The water is allowed to escape for instance by applying the load through a porous slab. When all the excess water has been squeezed out the axial strain is given by relations (2.11) in which we put $\sigma = 0$. We write

$$e_z = -p_0 a. \tag{3.7}$$

The coefficient

$$a = \frac{1-2\nu}{2G(1-\nu)} \tag{3.8}$$

will be called the *final compressibility*.

If we measure the axial strain just after the load has been applied so that the water has not had time to flow out, we must put $\theta = 0$ in relation (2.12). We deduce the value of the water pressure

$$\sigma = -\alpha Q e_z. \tag{3.9}$$

substituting this value in (2.11) we write

$$e_z = -p_0 a_i. \tag{3.10}$$

The coefficient

$$a_i = \frac{a}{1+\alpha^2 a Q} \tag{3.11}$$

will be called the *instantaneous compressibility*.

The physical constants considered above refer to the properties of the soil for the state of equilibrium when the water pressure is uniform throughout. We shall see hereafter that in order to study the transient state we must add to the four distinct constants above the so-called *coefficient of permeability* of the soil.

4. General Equations Governing Consolidation

We now proceed to establish the differential equations for the transient phenomenon of consolidation, i.e., those equations governing the distribution of stress, water content, and settlement as a function of time in a soil under given loads.

Substituting expression (2.11) for the stresses into the equilibrium conditions (1.2) we find

$$\begin{aligned} G\nabla^2 u + \frac{G}{1-2\nu}\frac{\partial \epsilon}{\partial x} - \alpha\frac{\partial \sigma}{\partial x} &= 0, \\ G\nabla^2 v + \frac{G}{1-2\nu}\frac{\partial \epsilon}{\partial y} - \alpha\frac{\partial \sigma}{\partial y} &= 0, \\ G\nabla^2 w + \frac{G}{1-2\nu}\frac{\partial \epsilon}{\partial z} - \alpha\frac{\partial \sigma}{\partial z} &= 0, \end{aligned} \tag{4.1}$$

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2.$$

There are three equations with four unknowns u, v, w, σ. In order to have a complete system we need one more equation. This is done by introducing Darcy's law governing the flow of water in a porous medium. We consider again an elementary cube of soil and call V_x the volume of water flowing per second and unit area through the face of this cube perpendicular to the x axis. In the same way we define V_y and V_z. According to Darcy's law these three components of the rate of flow are related to the water pressure by the relations

$$V_x = -k\frac{\partial \sigma}{\partial x}, \quad V_y = -k\frac{\partial \sigma}{\partial y}, \quad V_z = -k\frac{\partial \sigma}{\partial z}. \tag{4.2}$$

The physical constant k is called the *coefficient of permeability* of the soil. On the other hand, if we assume the water to be incompressible the rate of water content of an element of soil must be equal to the volume of water entering per second through the surface of the element, hence

$$\frac{\partial \theta}{\partial t} = -\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} - \frac{\partial V_z}{\partial z}. \tag{4.3}$$

Combining Eqs. (2.$\overset{1}{\cancel{2}}$) (4.2) and (4.3) we obtain

$$k\nabla^2\sigma = \alpha\frac{\partial \epsilon}{\partial t} + \frac{1}{Q}\frac{\partial \sigma}{\partial t}. \tag{4.4}$$

The four differential Eqs. (4.1) and (4.4) are the basic equations satisfied by the four unknowns u, v, w, σ.

5. Application to a Standard Soil Test

Let us examine the particular case of a column of soil supporting a load $p_0 = -\sigma_z$ and confined laterally in a rigid sheath so that no lateral expansion can occur. It is assumed also that no water can escape laterally or through the bottom while it is free to escape at the upper surface by applying the load through a very porous slab.

Take the z axis positive downward; the only component of displacement in this case will be w. Both w and the water pressure σ will depend only on the coordinate z and the time t. The differential Eqs. (4.1) and (4.4) become

$$\frac{1}{a}\frac{\partial^2 w}{\partial z^2} - \alpha\frac{\partial \overset{\sigma}{\cancel{w}}}{\partial z} = 0, \tag{5.1}$$

$$k\frac{\partial^2 \sigma}{\partial z^2} = \alpha\frac{\partial^2 w}{\partial z \partial t} + \frac{1}{Q}\frac{\partial \sigma}{\partial t}, \tag{5.2}$$

where a is the final compressibility defined by (3.8). The stress σ_z throughout the loaded column is a constant. From (2.11) we have

$$p_0 = -\sigma_z = -\frac{1}{a}\frac{\partial w}{\partial z} + \alpha\sigma \tag{5.3}$$

and from (2.12)

$$\theta = \alpha\frac{\partial w}{\partial z} + \frac{\sigma}{Q}.$$

Note that Eq. (5.3) implies (5.1) and that

$$\frac{1}{a}\frac{\partial^2 w}{\partial z \partial t} = \alpha\frac{\partial \sigma}{\partial t}.$$

This relation carried into (5.2) gives

$$\frac{\partial^2 \sigma}{\partial z^2} = \frac{1}{c}\frac{\partial \sigma}{\partial t}, \tag{5.4}$$

with

$$\frac{1}{c} = \alpha^2\frac{a}{k} + \frac{1}{Qk}. \tag{5.5}$$

The constant c is called the *consolidation constant*. Equation (5.4) shows the important result that the water pressure satisfies the well-known equation of heat conduction. This equation along with the boundary and the initial conditions leads to a complete solution of the problem of consolidation.

Taking the height of the soil column to be h and $z=0$ at the top we have the boundary conditions

$$\begin{aligned} &\sigma = 0 \quad \text{for } z=0, \\ &\frac{\partial \sigma}{\partial z} = 0 \quad \text{for } z=h. \end{aligned} \tag{5.6}$$

The first condition expresses that the pressure of the water under the load is zero because the permeability of the slab through which the load is applied is assumed to be large with respect to that of the soil. The second condition expresses that no water escapes through the bottom.

The initial condition is that the change of water content is zero when the load is applied because the water must escape with a finite velocity. Hence from (2.12)

$$\theta = \alpha\frac{\partial w}{\partial z} + \frac{\sigma}{Q} = 0 \quad \text{for } t=0.$$

Carrying this into (5.3) we derive the initial value of the water pressure

$$\sigma = p_0 \Big/ \left(\frac{1}{\alpha a Q} + \alpha\right) \quad \text{for } t=0 \quad \text{or} \quad \sigma = \frac{a-a_i}{\alpha a} p_0, \tag{5.7}$$

where a_i and a are the instantaneous and final compressibility coefficients defined by (3.8) and (3.11).

The solution of the differential equation (5.4) with the boundary conditions (5.6) and the initial condition (5.7) may be written in the form of a series

$$\sigma = \frac{4}{\pi}\frac{a-a_i}{\alpha a} p_0 \left\{ \exp\left[-\left(\frac{\pi}{2h}\right)^2 ct\right] \sin\frac{\pi z}{2h} + \frac{1}{3}\exp\left[-\left(\frac{3\pi}{2h}\right)^2 ct\right] \sin\frac{3\pi z}{2h} + \cdots \right\}. \tag{5.8}$$

The settlement may be found from relation (5.3). We have

$$\frac{\partial w}{\partial z} = \alpha a \sigma - a p_0. \tag{5.9}$$

The total settlement is

$$w_0=-\int_0^h \frac{\partial w}{\partial z}dz=-\frac{8}{\pi^2}(a-a_i)hp_0\sum_0^\infty \frac{1}{(2n+1)^2}\exp\left\{-\left[\frac{(2n+1)\pi}{2h}\right]^2 ct\right\}+ahp_0. \tag{5.10}$$

Immediately after loading ($t=0$), the deflection is

$$w_i=-\frac{8}{\pi^2}(a-a_i)hp_0\sum_0^\infty \frac{1}{(2n+1)^2}+ahp.$$

Taking into account that

$$\sum_0^\infty \frac{1}{(2n+1)^2}=\frac{\pi^2}{8},\quad w_i=a_ihp_0, \tag{5.11}$$

which checks with the result (3.10) above. The final deflection for $t=\infty$ is

$$w_\infty=ahp_0. \tag{5.12}$$

It is of interest to find a simplified expression for the law of settlement in the period of time immediately after loading. To do this we first eliminate the initial deflection w_i by considering

$$w_s=w_0-w_i=\frac{8}{\pi^2}(a-a_i)hp_0\sum_0^\infty \frac{1}{(2n+1)^2}\left\{1-\exp\left[-\left(\frac{(2n+1)\pi}{2h}\right)^2 ct\right]\right\}. \tag{5.13}$$

This expresses that part of the deflection which is caused by consolidation. We then consider the rate of settlement.

$$\frac{dw_s}{dt}=\frac{2c(a-a_i)}{h}p_0\sum_0^\infty \exp\left\{-\left[\frac{(2n+1)\pi}{2h}\right]^2 ct\right\}. \tag{5.14}$$

For $t=0$ this series does not converge; which means that at the first instant of loading the rate of settlement is infinite. Hence the curve representing the settlement w_s as a function of time starts with a vertical slope and tends asymptotically toward the value $(a-a_i)hp_0$ as shown in Fig. 1 (curve 1). It is obvious that during the initial period of settlement the height h of the column cannot have any influence on the phenomenon because the water pressure at the depth $z=h$ has not yet had time to change. Therefore in order to find the nature of the settlement curve in the vicinity of $t=0$ it is enough to consider the case where $h=\infty$. In this case we put

$$n/h=\xi,\quad 1/h=\Delta\xi$$

and write (5.14) as

$$\frac{dw_s}{dt}=2c(a-a_i)p_0\sum_0^\infty \exp\left[-\pi^2(\xi+\tfrac{1}{2}\Delta\xi)^2ct\right]\Delta\xi$$

for $h=\infty$. The rate of settlement becomes the integral

$$\frac{dw_s}{dt}=2c(a-a_i)p_0\int_0^\infty \exp\left(-\pi^2\xi^2ct\right)d\xi=\frac{c(a-a_i)p_0}{(\pi ct)^{\frac{1}{2}}}. \tag{5.15}$$

The value of the settlement is obtained by integration

$$w_s=\int_0^t \frac{dw_s}{dt}dt=2(a-a_i)p_0\left(\frac{ct}{\pi}\right)^{\frac{1}{2}}. \tag{5.16}$$

It follows a parabolic curve as a function of time (curve 2 in Fig. 1).

6. Simplified Theory for a Saturated Clay

For a completely saturated clay the standard test shows that the initial compressibility a_i may be taken equal to zero compared to the final compressibility a, and that the volume change of the soil is equal to the amount of water squeezed out. According to (2.12) and (3.11) this implies

$$Q=\infty, \quad \alpha=1. \tag{6.1}$$

This reduces the number of physical constants of the soil to the two elastic constants and the permeability. From relations (3.5) and (3.6) we deduce

$$H=R=\frac{2G(1+\nu)}{3(1-2\nu)} \tag{6.2}$$

and from (5.5) the value of the consolidation constant takes the simple form

$$c=k/a. \tag{6.3}$$

Relation (2.12) becomes

$$\theta=\epsilon. \tag{6.4}$$

The general differential equations (4.1) and (4.4) are simplified,

$$\begin{aligned} G\nabla^2 u+\frac{G}{1-2\nu}\frac{\partial\epsilon}{\partial x}-\frac{\partial\sigma}{\partial x}&=0,\\ G\nabla^2 v+\frac{G}{1-2\nu}\frac{\partial\epsilon}{\partial y}-\frac{\partial\sigma}{\partial y}&=0,\\ G\nabla^2 w+\frac{G}{1-2\nu}\frac{\partial\epsilon}{\partial z}-\frac{\partial\sigma}{\partial z}&=0, \end{aligned} \tag{6.5}$$

$$k\nabla\sigma^2=\frac{\partial\epsilon}{\partial t}. \tag{6.6}$$

By adding the derivatives with respect to x, y, z of Eqs. (6.5), respectively, we find

$$\nabla\epsilon^2=a\nabla\sigma^2, \tag{6.7}$$

where a is the final compressibility given by (3.8).

From (6.6) and (6.7) we derive

$$\nabla\epsilon^2=\frac{1}{c}\frac{\partial\epsilon}{\partial t}. \tag{6.8}$$

Hence the volume change of the soil satisfies the equation of heat conduction.

Equations (6.5) and (6.8) are the fundamental equations governing the consolidation of a completely saturated clay. Because of (6.4) the initial condition $\theta=0$ becomes $\epsilon=0$, i.e., at the instant of loading no volume change of the soil occurs. This condition introduced in Eq. (6.7) shows that at the instant of loading the water pressure in the pores also satisfies Laplace's equation.

$$\nabla\sigma^2=0. \tag{6.9}$$

The settlement for the standard test of a column of clay of height h under the load p_0 is given by (5.13) by putting $a_i=0$.

$$w_s=\frac{8}{\pi^2}ahp_0\sum_0^\infty\frac{1}{(2n+1)^2}\times\left\{1-\exp\left[-\left(\frac{(2n+1)\pi}{2h}\right)^2 ct\right]\right\}. \tag{6.10}$$

From (5.16) the settlement for an infinitely high column is

$$w_s=2ap_0\left(\frac{ct}{\pi}\right)^{\frac{1}{2}}. \tag{6.11}$$

It is easy to imagine a mechanical model having the properties implied in these equations. Consider a system made of a great number of small rigid particles held together by tiny helical springs. This system will be elastically deformable and will possess average elastic constants. If we fill completely with water the voids between the

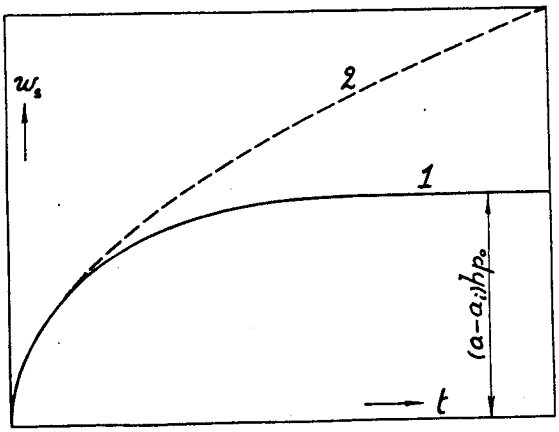


Fig. 1. Settlement caused by consolidation as a function of time. Curve 1 represents the settlement of a column of height h under a load p_0. Curve 2 represents the settlement for an infinitely high column.

particles, we shall have a model of a completely saturated clay.

Obviously such a system is incompressible if no water is allowed to be squeezed out (this corresponds to the condition $Q=\infty$) and the change of volume is equal to the volume of water squeezed out (this corresponds to the condition $\alpha=1$). If the systems contained air bubbles this would not be the case and we would have to consider the general case where Q is finite and $\alpha\neq 1$.

Whether this model represents schematically the actual constitution of soils is uncertain. It is quite possible, however, that the soil particles are held together by capillary forces which behave in pretty much the same way as the springs of the model.

7. Operational Calculus Applied to Consolidation

The calculation of settlement under a suddenly applied load leads naturally to the application of operational methods, developed by Heaviside for the analysis of transients in electric circuits. As an illustration of the power and simplicity introduced by the operational calculus in the treatment of consolidation problem we shall derive by this procedure the settlement of a completely saturated clay column already calculated in the previous section. In subsequent articles the operational method will be used extensively for the solution of various consolidation problems. We consider the case of a clay column infinitely high and take as before the top to be the origin of the vertical coordinate z. For a completely saturated clay $\alpha=1$, $Q=\infty$ and with the operational notations, replacing $\partial/\partial t$ by p, Eqs. (5.1) become

$$\frac{1}{a}\frac{\partial^2 w}{\partial z^2}=\frac{\partial\sigma}{\partial z},\quad k\frac{\partial^2\sigma}{\partial z^2}=p\frac{\partial w}{\partial z}. \tag{7.1}$$

A solution of these equations which vanishes at infinity is

$$\begin{aligned} w&=C_1e^{-z(p/c)^{\frac{1}{2}}},\\ \sigma&=C_2-\frac{1}{a}\left(\frac{p}{c}\right)^{\frac{1}{2}}C_1e^{-z(p/c)^{\frac{1}{2}}}. \end{aligned} \tag{7.2}$$

The boundary conditions are for $z=0$

$$\sigma_z=-1=\frac{1}{a}\frac{\partial w}{\partial z},\quad \sigma=0.$$

Hence

$$C_1=a\left(\frac{c}{p}\right)^{\frac{1}{2}},\quad C_2=1.$$

The settlement w_s at the top ($z=0$) caused by the sudden application of a unit load is

$$w_s=a\left(\frac{c}{p}\right)^{\frac{1}{2}}\cdot 1(t).$$

The meaning of this symbolic expression is derived from the operational equation[4]

$$\frac{1}{p^{\frac{1}{2}}}1(t)=2\left(\frac{t}{\pi}\right)^{\frac{1}{2}}. \tag{7.3}$$

The settlement as a function of time under the load p_0 is therefore

$$w_s=2ap_0\left(\frac{ct}{\pi}\right)^{\frac{1}{2}}. \tag{7.4}$$

This coincides with the value (6.11) above.

[4] V. Bush, *Operational Circuit Analysis* (John Wiley, New York, 1929), p. 192.

BIOT THEORY OF CONSOLIDATION

J. Appl. Phys. (1941) p155 - 164

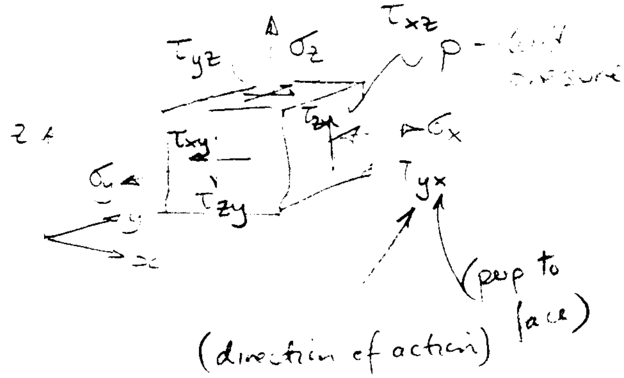
General 3-D Theory to augment 1-D theory of Terzaghi (1923)

Assumptions

1. Isotropic, linearly elastic, reversible material (reversible strains)
2. Small strains
3. Darcy's law applies

Assumes differential element is large compared to no. of grains and contains a statistical average – treat as stresses rather than non-uniform intergranular forces

1. Soil stresses



Assume tensile stresses } +ve
delta.. }

compressive pore pressures +ve

Stress components: $\underline{\sigma}^T = \{ \sigma_x , \sigma_y , \sigma_z , \tau_{xy} , \tau_{xz} , \tau_{yz} \}$ (1.1)

Equilibrium equation (static - quasi static)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = b_x - \rho \frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = b_y - \rho \frac{\partial^2 u_y}{\partial t^2} \quad (1.2)$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = b_z - \rho \frac{\partial^2 u_z}{\partial t^2}$$

Equilibrium equations are written for total stresses that include intergranular stresses and pore fluid pressure.

1. RESULTING STRAINS

Define displacements $[u_x, u_y, u_z]$

Strains $\underline{\varepsilon}^T = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}]$

Assuming infinitesimal strains for solid body

$$\left.\begin{aligned}
\varepsilon_x &= \frac{\partial u_x}{\partial x} & \gamma_{xy} &= \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\
\varepsilon_y &= \frac{\partial u_y}{\partial y} & \gamma_{yz} &= \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\
\varepsilon_z &= \frac{\partial u_z}{\partial z} & \gamma_{zx} &= \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)
\end{aligned}\right\} \quad (2.1)$$

Fluid strains to define fluid volume in pores

θ = change in fluid volume per unit volume of soil.

p = pore fluid pressure (in pores)

- Assume strains are uniform throughout differential volume.
- No pore pressure gradients.

Assuming reversibility then 7 variables

$$\varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx} \quad \theta$$

are (unique) functions of

$$\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx} \quad p$$

Assume $p = 0$ and define Hooke's law

$$\left.\begin{aligned}
\varepsilon_x &= \tfrac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \\
\varepsilon_y &= \tfrac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \\
\varepsilon_z &= \tfrac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \\
\gamma_{xy} &= \tau_{xy}/G \\
\gamma_{yz} &= \tau_{yz}/G \\
\gamma_{zx} &= \tau_{zx}/G
\end{aligned}\right\} \quad (2.2)$$

$$G = \frac{E}{2(1+\nu)} \qquad \text{(shear modulus)}$$

Adding pore fluid pressure, p

- produces no shear strain
- isotropic influence on all normal strain components

From (2.2)

$$\begin{aligned}
\varepsilon_x &= \tfrac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] + \frac{p}{3H} \\
\varepsilon_y &= \tfrac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] + \frac{p}{3H} \\
\varepsilon_z &= \tfrac{1}{E}\left[\sigma_z - \nu(\sigma_x - \sigma_y)\right] + \frac{p}{3H} \qquad (2.3)
\end{aligned}$$

$$\left.\begin{aligned}
\gamma_{xy} &= \tau_{xy}/G \\
\gamma_{yz} &= \tau_{yz}/G \\
\gamma_{zx} &= \tau_{zx}/G
\end{aligned}\right\} \quad \text{no shear strain due to } p = 0$$

H is an arbitrary constant

Water content is also controlled by the seven stresses. In general

$$\theta = a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z + a_4 \tau_{xy} + a_5 \tau_{yz} + a_6 \tau_{zx} + a_7 p \qquad (2.4)$$

Shear strain causes no net volume change $\therefore$ no effect on θ

hence $\quad a_4 = a_5 = a_6 = 0$

Since isotropic $\quad a_1 = a_2 = a_3 \neq 0$

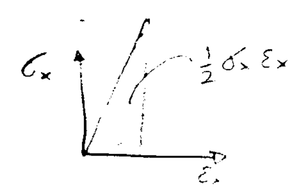
$\therefore$ Rewrite (2.4) as

$$\boxed{\theta = \frac{1}{3H_1}(\sigma_x + \sigma_y + \sigma_z) + \frac{p}{R}} \qquad (2.5)$$

arbitrary constants.

So far in (2.3) and (2.5) <u>5 constants</u> E, ν, H, H_1, R

<u>Attempt to show $H = H_1$</u>



Concept of potential energy of soil, U.

$$U = \frac{1}{2}[\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx} + p\theta] \qquad (2.6)$$

Assume, for example

$$\left.\begin{aligned} \sigma_x &= \sigma_y = \sigma_z = \sigma \\ \tau_{xy} &= \tau_{yz} = \tau_{zx} = 0 \end{aligned}\right\} \qquad (2.7)$$

volumetric strain

Potential energy: $$\mathcal{U} = \frac{1}{2}(\sigma \varepsilon_v + p\theta) \qquad (2.8)$$

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

From Hooke's law of (2.3) $$\boxed{\varepsilon_v = \frac{3}{E}(1-2\nu)\sigma + \frac{p}{H}} \qquad (2.9)$$

and volumetric strain constitutive relation of (2.5)

$$\theta = \frac{\sigma}{H_1} + \frac{p}{R} \qquad (2.10)$$

Solve (2.9) and (2.10) for σ and p, rewrite as

$$\begin{bmatrix} \frac{3(1-2\nu)}{E} & \frac{1}{H} \\ \frac{1}{H_1} & \frac{1}{R} \end{bmatrix} \begin{Bmatrix} \sigma \\ p \end{Bmatrix} = \begin{Bmatrix} \varepsilon_v \\ \theta \end{Bmatrix} \qquad (2.11)$$

Invert

$$\frac{1}{\frac{3(1-2\nu)}{ER} - \frac{1}{H_1 H}} \begin{bmatrix} \frac{1}{R} & -\frac{1}{H} \\ -\frac{1}{H_1} & \frac{3(1-2\nu)}{E} \end{bmatrix} \begin{Bmatrix} \varepsilon_v \\ \theta \end{Bmatrix} = \begin{Bmatrix} \sigma \\ p \end{Bmatrix} \qquad (2.12)$$

OR

$$\sigma = \frac{\varepsilon_v}{R\Delta} - \frac{\theta}{H\Delta} \qquad (2.13)$$

$$p = \frac{-\varepsilon_v}{H_1\Delta} + \frac{3(1-2\nu)\theta}{E\Delta} \qquad (2.14)$$

with

$$\Delta = \frac{3(1-2\nu)}{ER} - \frac{1}{H_1 H} \qquad (2.15)$$

Potential energy is given by the variables ε_v, θ from (2.8)

$$U = \tfrac{1}{2}(\sigma\varepsilon_v + p\theta)$$

$$\frac{\partial u}{\partial \varepsilon_v} = \frac{1}{2}\sigma \quad ; \quad \frac{\partial u}{\partial \theta} = \frac{1}{2}p \qquad (2.16)$$

Now, differentiating w.r.t $\partial/\partial\theta$ and $\partial/\partial\varepsilon_v$, respectively

$$\underbrace{\frac{\partial}{\partial\theta}\frac{\partial(u)}{\partial\varepsilon_v}} = \frac{1}{2}\frac{\partial\sigma}{\partial\theta} \quad ; \quad \underbrace{\frac{\partial}{\partial\varepsilon_v}\frac{\partial(u)}{\partial\theta}} = \frac{1}{2}\frac{\partial p}{\partial\varepsilon_v} \qquad (2.17)$$

Equating,

$$\frac{\partial\sigma}{\partial\theta} = \frac{\partial p}{\partial\varepsilon_v} \qquad (2.18)$$

Returning to equation (2.13) $\quad \dfrac{\partial(\sigma)}{\partial\theta} = -\dfrac{1}{H\Delta}$

equation (2.14) $\quad \dfrac{\partial(p)}{\partial\varepsilon_v} = -\dfrac{1}{H_1\Delta}$

$$\left.\right\} \frac{1}{H\Delta} = \frac{1}{H_1\Delta} \qquad (2.19)$$

Therefore proven that $H = H_1$ and therefore only 4 independent constants; E, ν, H, R

Hence equation (2.5) is restated as

$$\theta = \frac{1}{3H}(\sigma_x + \sigma_y + \sigma_z) + \frac{P}{R} \qquad (2.20)$$

This and equations (2.3) describe the equilibrium conditions for strain and moisture content.

The inverse equations of (2.3) may be rewritten as

Create inverse relations: $\varepsilon = f(\sigma, p) \longrightarrow \sigma = f(\varepsilon, p)$

$\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z$

Define volume strain:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \varepsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \varepsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned}\right\} \quad \begin{aligned} \varepsilon_v &= \frac{1}{E}[3\sigma_m - 2\nu(3\sigma_m)] \\ \varepsilon_v &= \frac{3}{E}[\sigma_m - 2\nu\sigma_m] \end{aligned}$$

$$\varepsilon_v = \frac{3\sigma_m}{E}[1 - 2\nu]$$

$$\sigma_m = \frac{\varepsilon_v E}{3(1-2\nu)}$$

Rewrite Hooke's Law:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_x = \frac{1}{E}[\sigma_x + \nu\sigma_x - \nu\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_x = \frac{1}{E}[(1+\nu)\sigma_x - \nu(\sigma_x + \sigma_y + \sigma_z)] = \frac{1}{E}[(1+\nu)\sigma_x - \nu(3\sigma_m)]$$

Rearrange in terms of σ_x:

Substitute for σ_m

$$\frac{\varepsilon_x E}{(1+\nu)} + \frac{\nu 3\sigma_m}{(1+\nu)} = \sigma_x$$

$$G = \frac{E}{2(1+\nu)}$$

$$\frac{2}{2}\frac{\varepsilon_x E}{(1+\nu)} + \frac{2}{2}\frac{3\nu}{(1+\nu)}\frac{\varepsilon_v E}{3(1-2\nu)} = \sigma_x$$

$$2G\,\varepsilon_x + 2G\frac{\nu\,\varepsilon_v}{(1-2\nu)} = \sigma_x$$

Inverse strain relations of eqn (2.3)

$$\left.\begin{aligned}
\sigma_x &= 2G\left(\varepsilon_x + \frac{\nu\varepsilon_v}{1-2\nu}\right) - \alpha p \\
\sigma_y &= 2G\left(\varepsilon_y + \frac{\nu\varepsilon_v}{1-2\nu}\right) - \alpha p \\
\sigma_z &= 2G\left(\varepsilon_z + \frac{\nu\varepsilon_v}{1-2\nu}\right) - \alpha p \\
\tau_{xy} &= G\gamma_{xy} \quad ; \quad \tau_{xz} = G\gamma_{xz} \quad ; \quad \tau_{yz} = G\gamma_{yz}
\end{aligned}\right\} \quad (2.21)$$

with $\alpha = \dfrac{2(1+\nu)}{3(1-2\nu)}\dfrac{G}{H} = \dfrac{1}{3(1-2\nu)}\dfrac{E}{H}$ (2.22)

Also the relationship $\theta = \frac{1}{3H}(\sigma_x + \sigma_y + \sigma_z) + \frac{P}{R}$ of (2.20

may be resubstituted using (2.21) to give

$$\theta = \frac{1}{3H}\left\{2G\underbrace{(\varepsilon_x + \varepsilon_y + \varepsilon_z)}_{\varepsilon_v} + 2G\frac{3\nu\varepsilon_v}{(1-2\nu)} - 3\alpha p\right) + \frac{P}{R}$$

$$\theta = \frac{2G}{3H}\left\{1 + \frac{3\nu}{(1-2\nu)}\right\}\varepsilon_v - \frac{3\alpha p}{3H} + \frac{P}{R}$$

$$\theta = \frac{2G}{3H}\left\{\frac{1-2\nu+3\nu}{(1-2\nu)}\right\}\varepsilon_v - \left(\frac{\alpha}{H} - \frac{1}{R}\right)P$$

$$\theta = \frac{2G}{3H}\frac{(1+\nu)}{(1-2\nu)}\varepsilon_v - \left(\frac{\alpha}{H} - \frac{1}{R}\right)P$$

OR $$\theta = \alpha\varepsilon_v + \frac{P}{Q} \quad (2.23)$$

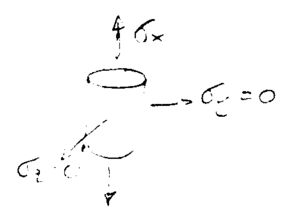
with $$\frac{1}{Q} = \frac{1}{R} - \frac{\alpha}{H} \qquad (2.24)$$

Summarizing
Four material constants E, ν, H, R

and subsidiary variables from these; G, Q, α

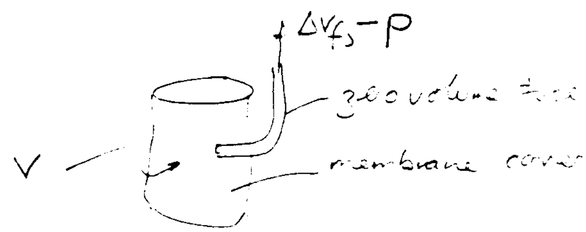
3. PHYSICAL INTERPRETATION OF PARAMETERS

E and ν are drained constants $\quad E = \frac{\sigma_x}{\varepsilon_x}$



$$\varepsilon_z = \varepsilon_y = \nu\varepsilon_x = \nu\frac{\sigma_x}{E}$$

Fluid parameters H, R



1) Suck out volume of fluid by applying pressure $-p$ at ΔV_f

Then $\theta = \frac{\Delta V_f}{V}$; no change in total stress

From (2.20) $$\theta = \frac{1}{3H}(\sigma_x + \sigma_y + \sigma_z) + \frac{p}{R}$$

$$R = \frac{p}{\theta}$$ R notes change in fluid strain with [illegible]

2) Volume change of soil is $\frac{\Delta V_s}{V} = \varepsilon_v$

From equn (2.3) $$\varepsilon_v = \frac{p}{H} \qquad H = \frac{p}{\varepsilon_v}$$

H notes change in soil volume [illegible]

Other appropriate constants

$$\alpha = \frac{2(1+\nu)}{3(1-2\nu)} \frac{G}{H}$$

from eqn (2.21) $\quad \theta = \alpha \varepsilon_v + \dfrac{P}{Q}$

drained test, $P=0$ $\quad \alpha = \dfrac{\theta}{\varepsilon_v}$ = ratio volume of fluid strain to solid strain under drained conditions.

And Q in (2.22) for $\quad \dfrac{1}{Q} = \dfrac{1}{R} - \dfrac{\alpha}{H}$

$\frac{1}{Q}$ represents amount of fluid that can be forced into the soil under pressure while volume remains constant ie compressibility of soil

ie since (2.21) $\quad \theta = \alpha \cancelto{0}{\varepsilon_v} - \dfrac{P}{Q}$

reflects compressibility of the fluid.

4. GENERAL EQUATIONS GOVERNING CONSOLIDATION

So far equations represent equilibrium state – no change in space or time.

Require to extend to spatial variation of dependent parameters.

Must satisfy
- Elastic solution
  - Equilibrium equn (1.2)
  - Constitutive equn (2.21)
  - Compatibility
  - Boundary conditions
- Flow equation
  - Continuity
  - Constitutive (Darcy's law)

<u>Elastic solution</u> – Written in terms of total stresses.

Substitute (2.21) into (1.2) to satisfy equilibrium and constitutive

First condition, for example:

$$\frac{\partial}{\partial x}\left\{2G\left(\frac{\partial u_x}{\partial x}+\frac{\nu\,\varepsilon_v}{(1-2\nu)}\right)-\alpha p\right\}+G\frac{\partial \overset{\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)}{\gamma_{xy}}}{\partial y}+G\frac{\partial \overset{\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)}{\gamma_{xz}}}{\partial z}=0 \qquad (4.1)$$

Separating coefficient

$$2G\frac{\partial^2 u_x}{\partial x^2}+G\frac{\partial^2 u_x}{\partial y^2}+G\frac{\partial^2 u_x}{\partial z^2}+G\frac{\partial^2 u_y}{\partial x\,\partial y}+G\frac{\partial^2 u_z}{\partial x\,\partial z}$$

$$+\,2G\frac{\nu}{(1-2\nu)}\frac{\partial \varepsilon_v}{\partial x}-\alpha\frac{\partial p}{\partial x}=0 \qquad (4.2)$$

$$G\left\{\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right\} + G\frac{\partial}{\partial x}\overbrace{\left\{\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right\}}^{\varepsilon_v}$$

$$+ \frac{2G\nu}{(1-2\nu)}\frac{\partial \varepsilon_v}{\partial x} - \alpha\frac{\partial p}{\partial x} = 0 \qquad (4.3)$$

$$G\nabla^2 u_x + \frac{2G\nu + G(1-2\nu)}{1-2\nu}\frac{\partial \varepsilon_v}{\partial x} - \alpha\frac{\partial p}{\partial x} = 0 \qquad (4.4)$$

$$\nabla^2 = \{\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2\}$$

And finally

$$G\nabla^2 u_x + \frac{G}{(1-2\nu)}\frac{\partial \varepsilon_v}{\partial x} - \alpha\frac{\partial p}{\partial x} = 0$$

Similarly for the other two equilibrium equations

$$\left.\begin{aligned} G\nabla^2 u_y + \frac{G}{(1-2\nu)}\frac{\partial \varepsilon_v}{\partial y} - \alpha\frac{\partial p}{\partial y} &= 0 \\ G\nabla^2 u_z + \frac{G}{(1-2\nu)}\frac{\partial \varepsilon_v}{\partial z} - \alpha\frac{\partial p}{\partial z} &= 0 \end{aligned}\right\} \qquad (4.5)$$

3 equations and 4 unknowns u_x, u_y, u_z, p

One more equation.

Flow Equation

Darcy's law

$$\left.\begin{aligned} v_x &= -k\frac{\partial p}{\partial x} \\ v_y &= -k\frac{\partial p}{\partial y} \\ v_z &= -k\frac{\partial p}{\partial z} \end{aligned}\right\} \qquad (4.6)$$

k = coefficient of permeability

Continuity equation

$$\frac{\partial \theta}{\partial t} = -\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} - \frac{\partial v_z}{\partial z} \qquad (4.7)$$

Substituting (4.6) and (2.21) into (4.7)

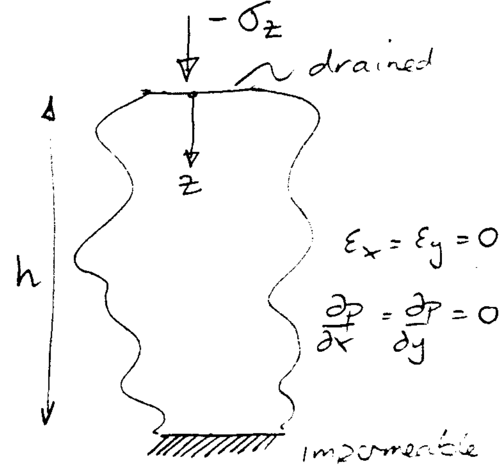
$$\theta = \alpha \varepsilon_v + \frac{p}{Q}$$

$$k\,\nabla^2 p = \frac{\partial}{\partial t}\left\{ \alpha \varepsilon_v + \frac{p}{Q} \right\} \qquad (4.8)$$

$$k\,\nabla^2 p = \alpha \frac{\partial \varepsilon_v}{\partial t} + \frac{1}{Q}\frac{\partial p}{\partial t} \qquad (4.9)$$

Equations (4.5) 3 eqns and (4.9)

4 equations in 4 unknowns $\longrightarrow$ solve



5. ONE-DIMENSIONAL CONSOLIDATION

- Apply load at top
- Compressible constituents.

One equation (4.5) represents equilibrium and constitutive relationship

$$G \nabla^2 u_z + \frac{G}{(1-2\nu)} \frac{\partial}{\partial z} (\cancel{\varepsilon_x}^{0} + \cancel{\varepsilon_y}^{0} + \varepsilon_z) - \alpha \frac{\partial p}{\partial z} = 0 \qquad (5.1)$$

Only remaining variables: u_z and p

$$\frac{G(1-2\nu)}{(1-2\nu)} \frac{\partial^2 u_z}{\partial z^2} + \frac{G}{(1-2\nu)} \frac{\partial}{\partial z} \frac{\partial u_z}{\partial z} - \alpha \frac{\partial p}{\partial z} = 0 \qquad (5.2)$$

$$\frac{2G(1-\nu)}{(1-2\nu)} \frac{\partial^2 u_z}{\partial z^2} - \alpha \frac{\partial p}{\partial z} = 0 \qquad (5.3)$$

Or using Biot terminology

$$\frac{1}{a} \frac{\partial^2 u_z}{\partial z^2} - \alpha \frac{\partial p}{\partial z} = 0 \qquad (5.4)$$

Flow equation from equation (4.9)

$$k \frac{\partial^2 p}{\partial z^2} = \alpha \frac{\partial}{\partial t} \frac{\partial u_z}{\partial z} + \frac{1}{Q} \frac{\partial p}{\partial t} \qquad (5.5)$$

From equation (2.21) the only appropriate constitutive equation in terms of the total stress applied at surface, $-\sigma_z$

$$-\sigma_z = -2G\left(\varepsilon_z + \frac{\nu}{1-2\nu}\varepsilon_v\right) + \alpha p \qquad (5.6)$$

$$-\sigma_z = -2G\left(\frac{\partial u_z}{\partial z}\frac{(1-2\nu)}{(1-2\nu)} + \frac{\nu}{(1-2\nu)}\frac{\partial u_z}{\partial z}\right) + \alpha p \qquad (5.7)$$

$$-\sigma_z = -2G\frac{(1-\nu)}{(1-2\nu)}\frac{\partial u_z}{\partial z} + \alpha p \qquad (5.8)$$

and finally
$$-\sigma_z = -\frac{1}{a}\frac{\partial u_z}{\partial z} + \alpha p \qquad (5.9)$$

to give pore pressure in the column

Alternatively from the volume strain of the fluid from (2.21)

$$\theta = \alpha\,\overset{\frac{\partial u_z}{\partial z}}{\varepsilon_v} + \frac{P}{Q} \qquad (5.10)$$

Note that dividing equation (5.9) by $\partial/\partial z$, gives

$$-\frac{\partial \sigma_z}{\partial z} = -\frac{1}{a}\frac{\partial^2 u_z}{\partial z^2} + \alpha\frac{\partial p}{\partial z} \qquad (5.11)$$

and comparing with (5.4) suggests that equilibrium eqn and Hooke's law are satisfied if $\partial\sigma_z/\partial z = 0$

This is also common sense.

Similarly with (5.9) and operating on with $\partial/\partial t$ gives

$$-\cancel{\frac{\partial \sigma_z}{\partial t}}^{\,0} = -\frac{1}{a}\frac{\partial^2 u_z}{\partial t\,\partial z} + \alpha\frac{\partial p}{\partial t} \qquad (5.12)$$

or

$$\frac{1}{a}\frac{\partial^2 u_z}{\partial t\,\partial z} = \alpha\frac{\partial p}{\partial t} \qquad (5.13)$$

Since equilibrium and Hooke's law satisfied by (5.4) and (5.11)
Substituting (5.13) into flow equation (5.5)

$$k\frac{\partial^2 p}{\partial z^2} = \alpha a \alpha\frac{\partial p}{\partial t} + \frac{1}{Q}\frac{\partial p}{\partial t} \qquad (5.14)$$

$$\boxed{\frac{\partial^2 p}{\partial z^2} = \left(\frac{\alpha^2 a}{k} + \frac{1}{Qk}\right)\frac{\partial p}{\partial t}} \qquad (5.15)$$

Consolidation equation — only one equation to solve
Use similar technique to Terzaghi (Heat flow)
Need initial pore pressures.

Initial conditions

$$\left.\begin{array}{ll} p = 0 & z = 0 \\ \dfrac{\partial p}{\partial z} = 0 & z = h \end{array}\right\} \qquad (5.16)$$

Initial pore fluid pressure, p_i $\qquad p_i \leq |{-\sigma_z}|$

Instantaneous load, zero volume change $\quad \theta = 0$

From (2.21) $\qquad \theta = \alpha\frac{\partial u_z}{\partial z} + \frac{p_i}{Q} = 0 \qquad (5.17)$

@ $t = 0$

$$\frac{\partial v_z}{\partial z} = -\frac{P_i}{Q\alpha} \qquad (5.18)$$

Substitute into equation (5.9)

$$-\sigma_z = -\frac{1}{a}\frac{\partial v_z}{\partial z} + \alpha p_i \qquad (5.19)$$

$$-\sigma_z = +\frac{1}{a}\frac{P_i}{Q\alpha} + \alpha p_i \qquad (5.20)$$

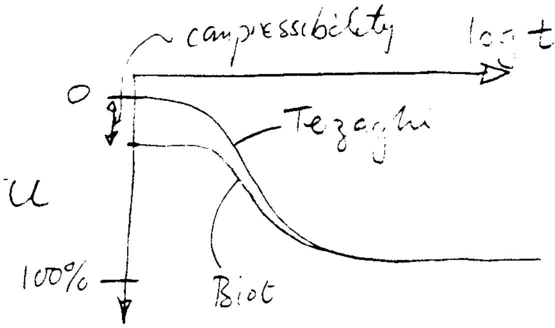
$$-\sigma_z = \left(\frac{1}{aQ\alpha} + \alpha\right) p_i \qquad (5.21)$$

and

$$p_i = -\sigma_z \Big/ \left(\frac{1}{aQ\alpha} + \alpha\right) \qquad (5.22)$$

Finally solve (5.15) subject to (5.16) and 5.22

Similar solution to Terzaghi by substituting p into displacement relation. Exactly mirrors except initial offset.



Initial offset since $p_i < |-\sigma_z|$

$\therefore$ compression of constituents with some load carried by effective stress

SUMMARY

EQUILIBRIUM

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = b_x - \rho \frac{\partial^2 u_x}{\partial t^2}$$

(1.2) 3 equns

$$\varepsilon_x = \frac{\partial u_x}{\partial x} \; ; \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \; ; \quad \theta = \Delta V_f / V$$

CONSTITUTIVE

$$\sigma_x = 2G\left(\varepsilon_x + \nu \frac{\varepsilon_v}{1-2\nu}\right) - \alpha p$$

(2.11) (3 eqn)

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] + \frac{p}{3H}$$

(2.4) (3 eqn)

$$\tau_{xy} = G\gamma_{xy}$$

(3 eqn)

$$\gamma_{xy} = \tau_{xy}/G$$

(3 eqn)

$$p = (\theta - \alpha \varepsilon_v)Q$$

(2.12) (1 eq)

$$\theta = \frac{1}{3H}(\sigma_x + \sigma_y + \sigma_z) + \frac{p}{R}$$

(1 eq)

(2.10)

$$\theta = \alpha \varepsilon_v + \frac{p}{Q}$$

(2.12)

Evaluate parameters: (E, ν, H, R)

E & ν from (2.4) $\longrightarrow G$

H & R from (2.10)

Then $$\alpha = \frac{2(1+\nu)}{3(1-2\nu)} \frac{G}{H} \; ; \quad \frac{1}{Q} = \frac{1}{R} - \frac{\alpha}{H}$$

FLOW

$$\theta = \alpha \varepsilon_v + \frac{p}{Q} \qquad v_x = -k\frac{dp}{dx}$$

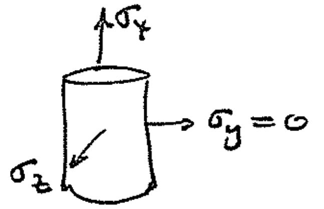
$$\frac{\partial \theta}{\partial t} = -\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} - \frac{\partial v_z}{\partial z}$$

Substitute (2.11) into (1.2) (3 equns) $$G\nabla^2 u_x + \frac{G}{(1-2\nu)}\frac{\partial \varepsilon_v}{\partial x} - \alpha \frac{\partial p}{\partial x} = 0$$

Substitute for flow: $$k\nabla^2 p = \frac{1}{Q}\frac{\partial p}{\partial t} + \alpha \frac{\partial \varepsilon_v}{\partial t}$$

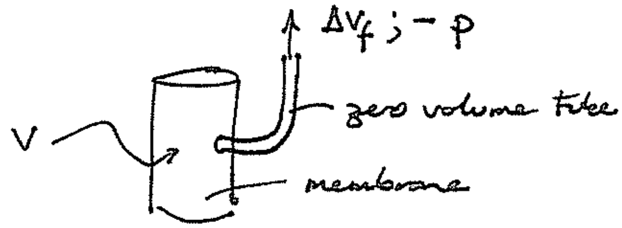
PHYSICAL INTERPRETATION OF PARAMETERS $\quad E, \nu, H, R \longrightarrow G, Q, \alpha$

Dry:



$$E = \frac{\sigma_x}{\varepsilon_x} \quad ; \quad \varepsilon_z = \varepsilon_y = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

Fluid filled + jacketed:



No change in total stress

1. Apply $-p$ and remove ΔV_f

$$\theta = \frac{\Delta V_f}{V} \; ; \quad \theta = \frac{1}{3H}\cancelto{0}{(\sigma_x + \sigma_y + \sigma_z)} + \frac{P}{R}$$

$$\therefore \quad R = P/\theta$$

(fluid strain with change in effective stress).

2. Measure ΔV_s (volume change of soil) for applied $-p$ and ΔV_f

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{3\cancelto{0}{\sigma_m}}{E}(1-2\nu) + \frac{P}{H} \qquad \therefore \quad H = \frac{P}{\varepsilon_v}$$

(solid strain with effective stress)

From: $\quad E, \nu, H, R$

$$G = E/2(1+\nu)$$

$$\alpha = \frac{2(1+\nu)}{3(1-2\nu)}\frac{G}{H}$$

$$\frac{1}{Q} = \frac{1}{R} - \frac{\alpha}{H}$$

4:3 Dual-porosity poroelasticity

EQUILIBRIUM EQUATION

Terzaghi effective stress

$$\partial \underline{\sigma}_1 = \partial \underline{\sigma}_1' + \underline{m}\, \partial p_1$$
$$\partial \underline{\sigma}_2 = \partial \underline{\sigma}_2' + \underline{m}\, \partial p_2 \qquad (1)$$

Equilibrium (local) $\quad \partial \underline{\sigma}_1 = \partial \underline{\sigma}_2 = \partial \underline{\sigma} \qquad (2)$

Constitutive
$$\partial \underline{\sigma}_1' = \underline{D}_1\, \partial \underline{\varepsilon}_1$$
$$\partial \underline{\sigma}_2' = \underline{D}_2\, \partial \underline{\varepsilon}_2 \qquad (3)$$

Inverse constitutive
$$\partial \underline{\varepsilon}_1 = \underline{C}_1\, \partial \underline{\sigma}_1' \qquad C_1 = D_1^{-1}$$
$$\partial \underline{\varepsilon}_2 = \underline{C}_2\, \partial \underline{\sigma}_2' \qquad C_2 = D_2^{-1} \qquad (4)$$

Total strain $\quad \partial \underline{\varepsilon} = \partial \underline{\varepsilon}_1 + \partial \underline{\varepsilon}_2 \qquad (5)$

Providing reference length includes phases (1) and (2)

Substituting (1) into (4) and then into (5)

$$\partial \underline{\varepsilon} = (\underline{C}_1 + \underline{C}_2)\, \partial \underline{\sigma} - \underline{C}_1 \underline{m}\, \partial p_1 - \underline{C}_2 \underline{m}\, \partial p_2$$
$$\partial \underline{\sigma} = \underline{D}_{12} (\partial \underline{\varepsilon} + \underline{C}_1 \underline{m}\, \partial p_1 + \underline{C}_2 \underline{m}\, \partial p_2) \qquad (6)$$

where $D_{12} = (C_1 + C_2)^{-1}$

(7)

Global Equilibrium (FE) $\quad \int \underline{B}^T \partial \underline{\sigma}\, dV - \partial \underline{f} = 0$

Where $\partial \underline{\varepsilon} = \underline{B}\, \partial u \qquad (8)$

$$\partial p_n = \underline{N}\, \partial \underline{p}_n$$

Substituting (6) and (8) into (7), and divide by Δt

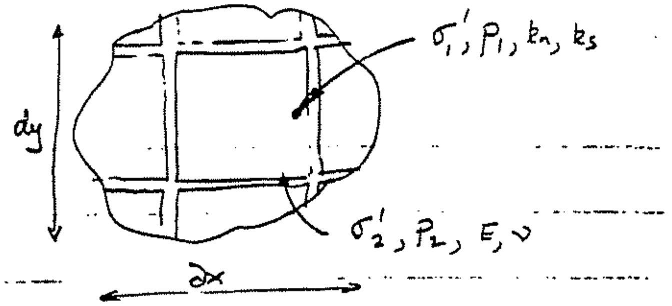
$$\int \underline{B}^T \underline{D}_{12} \underline{B}\, dV\, \dot{\underline{u}} + \int \underline{B}^T \underline{D}_{12} \underline{C}_1 \underline{m} \underline{N}\, dv\, \dot{\underline{p}}_1$$
$$+ \int \underline{B}^T \underline{D}_{12} \underline{C}_2 \underline{m} \underline{N}\, dv\, \dot{\underline{p}}_2 = \dot{\underline{f}} \qquad (9)$$

CONSERVATION OF MASS

Phase 1 - Constitutive $\quad \underline{v} = -\frac{k_1}{\mu} \underline{\nabla} (p_1 + \gamma z) \qquad (10)$

Continuity $\quad \underline{\nabla}^T \underline{v} = \underline{m}^T \dot{\underline{\varepsilon}} - \frac{n_1}{K_f} \dot{p}_1 + q_{12}$

DUAL POROSITY POROELASTICITY



Integrate using Green's Identity.

$$\frac{1}{\mu}\int \underline{A}^T k_1 \underline{A}\, dv\, \underline{p}_1 + \int \underline{N}^T \underline{m}^T \underline{C}_1 \underline{D}_{12} \underline{B}\, dv$$

$$+ \frac{n_1}{K_f}\int \underline{N}^T \underline{N}\, dv\, \dot{\underline{p}}_1 - \int \underline{N}^T \underline{N}\, dv\, \underline{q}_{12} = \frac{\gamma}{\mu}\int \underline{A}^T k_1 A d$$

(1

Similarly for phase 2.

$$\frac{1}{\mu}\int \underline{A}^T k_2 \underline{A}\, dv\, \underline{p}_2 + \int \underline{N}^T \underline{m}^T \underline{C}_2 \underline{D}_{12} \underline{B}\, dv\, \dot{u}$$

$$+ \frac{n_2}{K_f}\int \underline{N}^T \underline{N}\, dv\, \dot{\underline{p}}_2 - \int \underline{N}^T \underline{N}\, dv\, q_{12} = \frac{\gamma}{\mu}\int \underline{A}^T k_2 \underline{A}\, dv$$

(1

MATRIX EQUATIONS (COMBINED)

$$\begin{Bmatrix} 0 \\ \underline{K}_1 \underline{p}_1 \\ \underline{K}_2 \underline{p}_2 \end{Bmatrix}_{\ell+1} + \begin{bmatrix} F & G_1 & G_2 \\ E_1 & S_1 & 0 \\ E_2 & 0 & S_2 \end{bmatrix} \begin{Bmatrix} \dot{\underline{u}} \\ \dot{\underline{p}}_1 \\ \dot{\underline{p}}_2 \end{Bmatrix}_{\ell+1} = \begin{Bmatrix} \dot{f} \\ \underline{H} q_{12} + \underline{K}_1 \gamma z \\ \underline{H} q_{21} + \underline{K}_2 \gamma z \end{Bmatrix}_{\ell+1}$$

Fully implicit $\quad \dot{\underline{u}}^{\ell+1} = \frac{1}{\Delta t}(\underline{u}^{\ell+1} - \underline{u}^{\ell})$

etc. for $\dot{p}_1, \dot{p}_2$

gives $\quad \frac{1}{\Delta t} [\ ] \{\ \}^{\ell+1} = \frac{1}{\Delta t} [\ ] \{\ \}^{\ell} + \{\ \}$

solve

COMMENTS.

1. Form of $\underline{H}$ must be defined (Dual porosity
2. Displacements are lumped $\underline{u}$ not $\underline{u}_1, \underline{u}_2$

Flow-Deformation Response of Dual-Porosity Media

By Derek Elsworth,[1] Member, ASCE, and Mao Bai[2]

Abstract: A constitutive model is presented to define the linear poroelastic response of fissured media to determine the influence of dual porosity effects. A stress-strain relationship and two equations representing conservation of mass in the porous and fractured material are required. The behavior is defined in terms of the hydraulic and mechanical parameters for the intact porous matrix and the surrounding fracture system, allowing generated fluid pressure magnitudes and equilibration rates to be determined. Under undrained hydrostatic loading, the pore pressure–generation coefficients B, may exceed unity in either of the porous media or the fracture, representing a form of piston effect. Pressures generated within the fracture system equilibrate with time by reverse flow into the porous blocks. The equilibration time appears negligible for permeable sandstones, but it is significant for low-permeability geologic media. The constitutive model is represented in finite element format to allow solution for general boundary conditions where the influence of dual-porosity behavior may be examined in a global context.

Introduction

The linear flow-deformation behavior of geologic media is governed by the theory of single-porosity poroelasticity, as expounded originally by Biot (1941). Where, as in the case of fissured rock and soils, the medium comprises discrete fractions of differing solid compressibilities and permeabilities, a dual-porosity approach appears more appropriate.

The dual-porosity approach has been extensively developed to represent single-phase and multiphase flow in petroleum reservoirs. The original characterization of naturally fractured reservoirs by Warren and Root (1963) has been developed for radial flow in both blocky (Odeh 1965) and tabular reservoirs (Kazemi 1969) using analytical approaches and extended to multiphase flow using numerical techniques (Yamamoto et al. 1971; Kazemi et al. 1976; Kazemi and Merrill 1979; Thomas et al. 1983). Interest in single-phase behavior within dual-porosity reservoirs has been concerned with accurate representation of the flux fields within the porous and fractured components (Huyakorn et al. 1983) for application to mass (Bibby 1981) and thermal transport (Pruess and Narasimhan 1985; Elsworth 1989). In all of these applications it is assumed that total stresses remain constant with time, and therefore poroelastic effects are not incorporated.

The theory presented by Aifantis (1977, 1980) and Khaled et al. (1984) provides a suitable framework in which the flow-deformation behavior of dual-porosity media may be fully coupled to the deformation field as a multiphase continuum. Multiphase poroelasticity requires that the conditions of flow continuity within the different solid phases and the exchange between the phases are superimposed on the elastic displacement behavior of the fissured mass as body forces. As initially defined (Aifantis 1977), the constitutive coefficients describing the behavior of the aggregated medium

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[1]Associate Professor, Dept. of Mineral Engrg., Pennsylvania State Univ., University Park, PA 16802.

[2]Research Associate, Dept. of Mineral Engrg., Pennsylvania State Univ., University Park, PA 16802. Now at School of Petroleum and Geological Engineering, University of Oklahoma, Norman OK 73019-0628.

Note. Discussion open until June 1, 1992. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on May 11, 1989. This paper is part of the ***Journal of Geotechnical Engineering***, Vol. 118, No. 1, January, 1992. ©ASCE, ISSN 0733-9410/92/0001-0107/$1.00 + $.15 per page. Paper No. 26530.

defy direct physical interpretation. Although phenomenological coefficients describing both load-deformation and fluid-percolation response may be determined directly from laboratory and field testing of the fractured systems (Wilson and Aifantis 1982), these coefficients may be recovered more conveniently from basic knowledge of modulii and permeability of the components comprising matrix and fissure porosities.

Defining the response of the system directly in terms of the elastic and permeability properties of the components, with due regard for fissure geometry, offers the further advantage of ensuring that material nonlinearities of unknown magnitude are not inadvertently included within data reduced from field testing. This is an important factor, given long-standing knowledge on the nonlinear load-deformation behavior of interfaces (Goodman 1974) and the strong aperture dependence of fluid transmission in fissures (Iwai 1976). Indeed, the conditions needed to satisfy requirements for a linear theory of dual poroelasticity for fissured media may be so restrictive that, in all practicality, the phenomenon must be viewed as intrinsically nonlinear. This argument aside, only linear phenomena are considered in the following.

Where the structure of the fissured mass is well defined, as in the case of regularly jointed rocks and fissured soils, the contribution of fissure and matrix components to the overall flow and deformation response of the medium are readily apparent. Indeed, where elastic and flow properties of the fissures and matrix are known a priori, it is reasonable to develop the governing equations directly from this constituent basis. This exercise is completed in the following to develop the macroscopic continuum equations of linear poroelasticity on the basics of known fluid compressibility and defined fissure stiffness, porosities, and permeabilities.

Constitutive Equations

The behavior of the dual porosity aggregate may be defined in terms of component equations representing the solid deformation and coupled fluid-pressure response. The morphology of the continuum is represented in Fig. 1.

Solid Deformation

The relationship between changes in total stresses ($\partial\sigma$) and intergranular stresses ($\partial\sigma'$) are governed by the Terzaghi (1928, 1943) relationship of the form

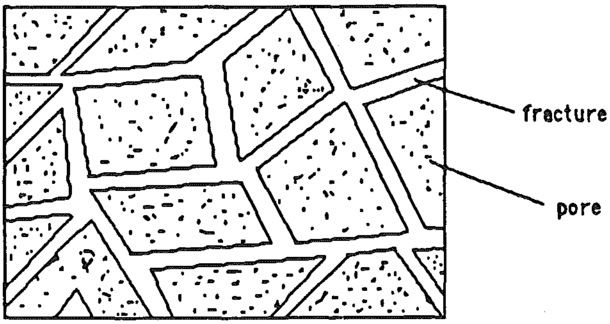


FIG. 1. Morphology of Porous-Fractured Aggregate

$$\partial \boldsymbol{\sigma}_1 = \partial \boldsymbol{\sigma}_1' + \mathbf{m}\partial p_1 \qquad (1a)$$

$$\partial \boldsymbol{\sigma}_2 = \partial \boldsymbol{\sigma}_2' + \mathbf{m}\partial p_2 \qquad (1b)$$

where ∂p = change in fluid pressure. The effect of grain compression on the intergranular stresses are neglected. Stresses and strains are positive in compression. Subscripts 1 and 2 refer to the porous and fractured phases, respectively, and are represented as $\partial \boldsymbol{\sigma}_1 = \partial[\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}]_1^T$ and $\partial \boldsymbol{\sigma}_1 = \partial[\sigma_{xx}, \sigma_{zz}, \sigma_{xz}]_1^T$ for three-dimensional and two-dimensional problems, respectively. Since pore fluid pressures act on the normal stresses alone, the vector $\mathbf{m}^T$ is [1,1,1,0,0,0] and [1,1,0] in three- and two-dimensions. Vector or matrix quantities are represented by boldface type. It may be readily noted that the influence of grain compression on the intergranular stress relationship (Skempton 1960; Nur and Byerlee 1971) [(1)] may be accommodated by substituting a modified magnitude of the vector **m**. This modified vector is determined as $\mathbf{m}_n = (\mathbf{m} - 1/3K_s\mathbf{D}_n\mathbf{m})$, where the subscript refers to the intergranular relation for the porous [$n = 1$ and (1*a*)] or fractured [$n = 2$ and (1*b*)] phases. In this, K_s is the bulk modulus of the solid and $\mathbf{D}_n$ is the elasticity matrix for phase n, as defined subsequently. The influence of grain compression on intergranular stresses are neglected because of the relative compressibility of the skeleton of interest in this study.

Local stress equilibrium requires that changes in total stress within adjacent phases must remain in equilibrium, such that

$$\partial \boldsymbol{\sigma}_1 = \partial \boldsymbol{\sigma}_2 = \partial \boldsymbol{\sigma} \qquad (2)$$

The linear constitutive relationships for the separate phases are defined as

$$\partial \boldsymbol{\sigma}_1' = \mathbf{D}_1 \partial \boldsymbol{\varepsilon}_1 \qquad (3a)$$

$$\partial \boldsymbol{\sigma}_2' = \mathbf{D}_2 \partial \boldsymbol{\varepsilon}_2 \qquad (3b)$$

where $\boldsymbol{\varepsilon}$ = solid strain within each of the two materials, namely porous and fractured material, and the inverse relations recovered from (3*a*,*b*) are

$$\partial \boldsymbol{\varepsilon}_1 = \mathbf{C}_1 \partial \boldsymbol{\sigma}_1' \qquad (4a)$$

$$\partial \boldsymbol{\varepsilon}_2 = \mathbf{C}_2 \partial \boldsymbol{\sigma}_2' \qquad (4b)$$

where $\mathbf{D}_n$ = the elasticity matrix for phase n with all other phases assumed rigid, and $\mathbf{C}_n$ = the compliance matrix under a similar arrangement, such that $\mathbf{D}_n^{-1} = \mathbf{C}_n$. Since $\mathbf{D}_n$ and $\mathbf{C}_n$ are written to represent the elastic behavior of phase n only, and are stated at a macroscopic level to contain a representative sample of matrix and fissure geometry, the total strain due to deformation in each of the phases is

$$\partial \boldsymbol{\varepsilon} = \partial \boldsymbol{\varepsilon}_1 + \partial \boldsymbol{\varepsilon}_2 \qquad (5)$$

or, substituting (1) into (4) and recording the result into (5) gives

$$\partial \boldsymbol{\varepsilon} = (\mathbf{C}_1 + \mathbf{C}_2)\partial \boldsymbol{\sigma} - \mathbf{C}_1\mathbf{m}\partial p_1 - \mathbf{C}_2\mathbf{m}\partial p_2 \qquad (6)$$

or

$$\partial \boldsymbol{\sigma} = \mathbf{D}_{12}(\partial \boldsymbol{\varepsilon} + \mathbf{C}_1\mathbf{m}\partial p_1 + \mathbf{C}_2\mathbf{m}\partial p_2) \qquad (7)$$

where $\mathbf{D}_{12} = (\mathbf{C}_1 + \mathbf{C}_2)^{-1}$ and dividing through by time increment ∂t allows (7) to be recorded as

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}_{12}(\dot{\boldsymbol{\varepsilon}} + \mathbf{C}_1\mathbf{m}\dot{p}_1 + \mathbf{C}_2\mathbf{m}\dot{p}_2) \quad (8)$$

representing the time-dependent load-deformation constitutive equation.

Fluid Pressure Response

Conservation of fluid mass must be maintained for each of the two porosity types, with appropriate transfer being possible between the two. The basic statement of continuity of flow requires that the divergence of the flow-velocity vector be equal to the rate of fluid accumulation per unit volume of space, i.e., $\boldsymbol{\nabla}^T\mathbf{v}$ = rate of accumulation. Equating the continuity constraint for the porous phase with changes in fluid mass due to all possible mechanisms gives the rate of fluid accumulation as the summation of: (1) Change in total body strain resulting in fluid expulsion; (2) change in fluid pressure precipitating changes in volumetric pore fluid content as a result of fluid and grain compressibility; and (3) volumetric transfer between the porous blocks and the fractures under differential pressures. These three source terms represent the right-hand side of the continuity relationship as

$$\boldsymbol{\nabla}^T\mathbf{v}_1 = \mathbf{m}^T\dot{\boldsymbol{\varepsilon}}_1 - \alpha_1\dot{p}_1 + K(p_1 - p_2) \quad (9)$$

where $\boldsymbol{\nabla}^T = (\partial/\partial x_1, \ldots, \partial/\partial x_i)$ in the case of i dimensional geometry; Eulerian flow velocity $\mathbf{v}_1^T = [v_{x1}, \ldots, v_{xi}]$ and $\alpha_1 = [n_1/K_f + (1 - n_1)/K_s]$, where n_1 = the porosity of phase 1; K_f = the fluid bulk modulus; and k_s = the solid grain bulk modulus. Transfer between the porous block and the surrounding fractures may be quantified by the assumption of a quasi-steady response governed by the coefficient K and the instantaneous pressure differential $(p_1 - p_2)$ (Warren and Root 1963).

In (9), the volume strain within the porous medium, $\mathbf{m}^T\boldsymbol{\varepsilon}_1$, refers to the drained condition where excess pore pressures are zero throughout the loading process. The proportion of the volume strain manifest within the porous phase may be determined from the total volume strain, $\mathbf{m}^T\boldsymbol{\varepsilon}_1$, by substituting (7) into (1) and the result into (4). This gives following division by ∂t, which

$$\mathbf{m}^T\dot{\boldsymbol{\varepsilon}}_1 = \mathbf{m}^T\mathbf{C}_1\mathbf{D}_{12}(\dot{\boldsymbol{\varepsilon}} + \mathbf{C}_1\mathbf{m}\dot{p}_1 + \mathbf{C}_2\mathbf{m}\dot{p}_2) - \mathbf{m}^T\mathbf{C}_1\mathbf{m}\dot{p}_1 \quad (10)$$

which may be reduced to the drained response by requiring that $\dot{p}_1 = \dot{p}_2 = 0$. [The requirement of using drained solid body strain may be confirmed by noting the form of (40) where pressure changes are set to $\dot{p}_1 = \dot{p}_2 = 0$ leaving the only mechanism for fluid expulsion into the separate phases through displacement rates, $\dot{\mathbf{u}}$.] The truncated (drained) form of (10) may then be substituted into the continuity relationship of (9) to yield

$$\boldsymbol{\nabla}^T\mathbf{v}_1 = \mathbf{m}^T\mathbf{C}_1\mathbf{D}_{12}\dot{\boldsymbol{\varepsilon}} - \alpha_1\dot{p}_1 - K(p_1 - p_2) \quad (11)$$

to give the final form of the continuity relationship for phase 1. Again, it is noted that the influence of grain compressibility as a result of changes in intergranular stresses has been neglected. This effect may, however, be incorporated by substituting the vector $\mathbf{m}_1$ for $\mathbf{m}$ in (11), where $\mathbf{m}_1$ accommodates the intergranular stress relationship of (1). This effect is small for the stress levels of interests within geotechnical engineering and may be neglected.

Repeating the same process for the fractured material gives a second continuity relationship

$$\boldsymbol{\nabla}^T\mathbf{v}_2 = \mathbf{m}^T\mathbf{C}_2\mathbf{D}_{12}\dot{\boldsymbol{\varepsilon}} - \alpha_2\dot{p}_2 + K(p_1 - p_2) \quad (12)$$

where the fracture compressibility term is defined as $\alpha_2 = n_2/K_f$, with n_2

representing the fracture porosity. Again the influence of grain compressibility on the intergranular stress relationship is neglected, but it may be readily included by substituting $\mathbf{m}_2$ for $\mathbf{m}$.

Dual-Porosity Load Response

For the case of general loading, the dual-porosity response may be fully described by (8), (11), and (12) to determine the magnitude of instantaneously generated displacements and fluid pressures and their modification with time. Although these relationships are entirely general, it is instructive to consider the behavior under purely hydrostatic loading and examine the response in normalized format. This will allow us to determine the important differences between true dual-porosity systems and their representation by an equivalent single-porosity system.

Hydrostatic Behavior

Where a hydrostatic load is applied to the dual-porosity medium, the three-dimensional form of (7) that contains a total of six subordinate equations may be reduced to a single component. For a ubiquitously jointed medium containing orthogonal fractures of uniform spacing, s, the strain components may be reduced on noting that

$$\mathbf{m}^T\varepsilon = 3\varepsilon \qquad (13)$$

and the compliance matrices are represented as

$$C_1 = \frac{(1 - 2\nu)}{E} \qquad (14a)$$

$$C_2 = \frac{1}{k_n s} \qquad (14b)$$

where k_n = the joint normal stiffness and the stiffness matrix reduces to

$$D_{12} = (C_1 + C_2)^{-1} \qquad (15)$$

Rearranging (8), (11), and (12) for an applied hydrostatic stress magnitude σ yields

$$\dot{\sigma} = D_{12}\dot{\varepsilon} + D_{12}C_1\dot{p}_1 + D_{12}C_2\dot{p}_2 \qquad (16)$$

$$K(p_1 - p_2) = 3D_{12}C_1\dot{\varepsilon} - \alpha_1\dot{p}_1 \qquad (17)$$

$$-K(p_1 - p_2) = 3D_{12}C_2\dot{\varepsilon} - \alpha_2\dot{p}_2 \qquad (18)$$

where a zero external flux condition ($\nabla^T\mathbf{v}_1 = \nabla^T\mathbf{v}_2 = 0$) has been applied as a further boundary condition. The behavior is most conveniently decomposed into: the undrained loading stage when fluid pressures p_{1_0} and p_{2_0} are generated; and the dissipation of pressures through fluid exchange in the subsequent drained behavior. The undrained pore pressures provide initial conditions to the initial value problem posed in (16)–(18). The magnitudes of undrained pore pressures may be evaluated, as discussed subsequently. Following undrained loading, dissipation begins under constant total stresses at the boundary where $\dot{\sigma} = 0$. Interest is restricted to the continuum level, where stress gradients are defined to be zero. Spatial changes in total stress that may occur within the volume, even under unchanged boundary stresses [see Mandel (1953)], are still admissible, but they require the additional step of incorporating the equilibrium relationship introduced in (34). These considerations are in addition to the constitutive equations and our interest, for the moment, is merely in the constitutive relations. With some rearrangement, the governing equations [(16)–(18)] may be represented as

$$\dot{\varepsilon} = -C_1\dot{p}_1 - C_2\dot{p}_2 \quad (19)$$

$$\dot{p}_1 = -\frac{\gamma_{11}}{\gamma_{12}}(p_1 - p_2) \quad (20)$$

$$\dot{p}_2 = \frac{\gamma_{22}}{\gamma_{12}}(p_1 - p_2) \quad (21)$$

where

$$\gamma_{11} = \frac{K}{3D_{12}}\left(\frac{\alpha_{22}}{C_1} + 1\right) \quad (22a)$$

$$\gamma_{22} = \frac{K}{3D_{12}}\left(\frac{\alpha_{11}}{C_2} + 1\right) \quad (22b)$$

$$\gamma_{12} = (\alpha_{11}\alpha_{22} - C_1C_2) \quad (22c)$$

$$\alpha_{11} = \left(\frac{\alpha_1}{3D_{12}C_1} + C_1\right) \quad (22d)$$

$$\alpha_{22} = \left(\frac{\alpha_2}{3D_{12}C_2} + C_2\right) \quad (22e)$$

Eqs. (20) and (21) are independent of the strain field and may be solved in time where boundary conditions are supplied directly. Solution to (20) and (21) may be determined as

$$p_1(t) = p_0 - (p_0 - p_{1_0})e^{-(\gamma_{11}+\gamma_{22})t/\gamma_{12}} \quad (23a)$$

$$p_2(t) = p_0 - (p_0 - p_{2_0})e^{-(\gamma_{11}+\gamma_{22})t/\gamma_{12}} \quad (23b)$$

where initial (undrained) conditions $p_1(0) = p_{1_0}$; $p_2(0) = p_{2_0}$ and terminal (fully drained) conditions $p_1(\infty) = p_2(\infty) = p_0$ control the response of the hydraulically closed system. The initial or undrained response occasioned upon application of a hydrostatic stress σ to the system may be evaluated by using the time-independent form of (16)–(18). Substituting the rearranged forms of (17) and (18) directly into (16) gives the instantaneous normalized strain ($D_{12}\varepsilon_0/\sigma$) as

$$\frac{D_{12}\varepsilon_0}{\sigma} = \left(1 + \frac{3D_{12}C_1^2}{\alpha_1} + \frac{3D_{12}C_2^2}{\alpha_2}\right)^{-1} = \frac{1}{\beta_1} \quad (24)$$

and the normalized pressure response is returned as

$$\frac{p_{1_0}}{\sigma} = \frac{3C_1}{\alpha_1\beta_1} = B_1 \quad (25)$$

$$\frac{p_{2_0}}{\sigma} = \frac{3C_2}{\alpha_2\beta_1} = B_2 \qquad (26)$$

where these also are equivalent to Skempton's pore pressure coefficients, B_1 and B_2, written separately for the fluid in the pores and fractures. With some rearrangement, the pore pressure parameters may be defined as

$$B_1 = \frac{1}{\dfrac{\alpha_1}{3C_1} + \dfrac{\left(1 + \dfrac{\alpha_1}{\alpha_2}\dfrac{C_2^2}{C_1^2}\right)}{\left(1 + \dfrac{C_2}{C_1}\right)}} \qquad (27)$$

and

$$B_2 = \frac{1}{\dfrac{\alpha_2}{3C_2} + \dfrac{\left(1 + \dfrac{\alpha_2}{\alpha_1}\dfrac{C_1^2}{C_2^2}\right)}{\left(1 + \dfrac{C_1}{C_2}\right)}} \qquad (28)$$

The pore pressure parameters are no longer bounded between zero and unity, as is the magnitude of the instantaneous normalized strain, which represents the aggregated influence of effective stresses in the two phases. A large magnitude of β_1 suggests that the material will exhibit a small instantaneous strain, although the fluid pressures generated are further controlled by the magnitude of the porous or fracture compliances and the compressibility of the fluid as embodied in α_1 and α_2.

The long-term response of the system where $\nabla^T\mathbf{v}_1 = \nabla^T\mathbf{v}_2 = 0$ results in an equilibrium pore pressure distribution with $p_1 = p_2$ and a steady magnitude of normalized strain. These long-term equilibrium parameters $[p_1(\infty)$, $p_2(\infty)$ and $\varepsilon(\infty)]$ are most easily recovered from solving the equivalent single-porosity problem, as follows.

Equivalent Single-Porosity Response

The equations developed to describe the dual-porosity response may be modified to represent the case where it is assumed that pressures within both the porous and fractured material remain in equilibrium (i.e., $p_1 = p_2$). This is the assumption made when real porous-fractured systems are represented by a simple equivalent phase system. Requiring $p_1 = p_2$ for all times, including initial pressure p_{1_0} and p_{2_0}, then (16) reduces to

$$\dot{\sigma} = D_{12}\dot{\varepsilon} + \dot{p} \qquad (29)$$

and adding (17) and (18) under a similar requirement that $\dot{p}_1 = \dot{p}_2$, gives

$$3\dot{\varepsilon} = (\alpha_1 + \alpha_2)\dot{p} \qquad (30)$$

where these represent the basic constitutive equations under hydrostatic loading. Implicit within (17) and (18), and by inference therefore in (30), is the requirement that $\nabla^T\mathbf{v} = 0$. Since no drainage is allowed, $\dot{\varepsilon}$ is zero following loading with the result that fluid pressures are maintained at their initially induced magnitude. This differs from the true dual-porosity case where there is an interchange of fluid between the pores and fractures. In the usual manner, (29) and (30) may be premultiplied by dt and the limit taken as $dt \rightarrow 0$ to recover the instantaneous normalized strain and pore pressure magnitudes as

$$\frac{D_{12}\varepsilon_0}{\sigma} = \left(1 + \frac{3}{D_{12}(\alpha_1 + \alpha_2)}\right)^{-1} = \frac{1}{\beta_2} \qquad (31)$$

and

$$\frac{P_0}{\sigma} = \frac{3}{D_{12}(\alpha_1 + \alpha_2)\beta_2} = \frac{1}{1 + \dfrac{D_{12}(\alpha_1 + \alpha_2)}{3}} = B \qquad (32)$$

where again B is bounded by zero and unity and incorporates the compliance of the porous solid, the fracture, and the fluid together with the appropriate distribution of porosities within each of the phases.

Parametric Response

Under undrained loading, the instantaneous generation of pore pressures is controlled by B_1 and B_2 in the true dual-porosity system, and by B_0 in the pseudo dual-porosity system. For dual porosity, the mismatch in medium stiffness and porosity between the porous body and the fracture result in differential pressure generation that will diffuse with time to an equilibrium configuration. Where external drainage is controlled, the equilibrium pressure is given by $p/\sigma = B_0$ where this magnitude is intermediate to the instantaneous pore and fracture pressures.

Of significance are the magnitudes of instantaneous pore pressures in the pore and fracture as controlled by B_1 and B_2 of (25) and (26). Unlike B [(32)] for the pseudo dual-porosity system, the magnitude of B_1 and B_2 are not confined between zero and unity. Rather, one may be greater than unity and the complementary parameter less than unity. This behavior is representative of a piston effect, whereby pressures are amplified in the low stiffness porosity. According to (25) and (26), B_i is largest for large α_i/α_j and small C_i/C_j for $i = 1, 2; j = 1, 2$ and $i \neq j$, and also is controlled by the ratio α_i/C_i. Physically, pore pressure magnitudes are increased as porosity increases or as solid stiffness decreases. The influence is illustrated for a variety of sedimentary and crystalline rocks in Table 1. Porosities and stiffnesses have been added to augment available data for poroelastic behavior of single-porosity solids.

Regardless of fracture spacing, all materials return B_2 magnitudes close to or greater than unity for the fracture pressures and B_1 magnitudes of negligible proportion. Pore pressure parameter magnitudes for the aggregated system given by B in the table remain close to unity. As the fracture spacing is increased (from 0.1 m to 0.5 m), the aggregated B decreases uniformly due to the net increase in stiffness of the system. However, no similar generalization is possible for the coefficients B_1 and B_2.

The instantaneous pore pressure regime in the closed system ($\nabla^T \cdot v = 0$) is modified with time by diffusive exchange between the porous body and the fracture. For realistic parameter estimates, given in Table 1, dif-

TABLE 1. Material Properties

Parameter (1)	Ruhr sandstone (2)	Tennessee marble (3)	Charcoal granite (4)	Berea sandstone (5)	Westerly granite (6)	Weber sandstone (7)
E(GPa)[a]	29.8	60.0	47.5	14.4	37.5	28.1
ν[a]	0.12	0.25	0.27	0.20	0.25	0.15
k_n(GPa · m^{-1})[b,d]	12.1	83.0	83.0	12.1	83.0	12.1
K_s(GPa)[a]	36.0	50.0	45.4	36.0	45.4	36.0
K_f(GPa)[a]	3.3	3.3	3.3	3.3	3.3	3.3
k_1(m)[b]	2×10^{-16}	1×10^{-19}	1×10^{-19}	1.9×10^{-13}	4×10^{-19}	1×10^{-15}
μ(GPa · s)	1×10^{-12}	1×10^{-12}	1×10^{-12}	1×10^{-12}	1×10^{-12}	1×10^{-12}
n_1[c]	0.032	0.0052	0.014	0.064	0.00106	0.176
n_2	0.0032[e]	0.052[f]	0.05	0.0064[e]	0.0106[f]	0.0176[e]
s(m)	0.1	0.1	0.1	0.1	0.1	0.1
B	0.986	0.912	0.905	0.982	0.941	0.969
B_1	8.5×10^{-4}	5.2×10^{-2}	4.9×10^{-2}	2.3×10^{-3}	1.9×10^{-2}	2.2×10^{-3}
B_2	1.031	1.018	1.029	1.050	1.104	1.028
t_{50}(s)	1.96×10^{-2}	2.08×10^{1}	2.44×10^{1}	2.40×10^{-5}	4.91×10^{0}	8.06×10^{-3}
t_{95}(s)	8.47×10^{-2}	8.98×10^{1}	1.05×10^{2}	1.06×10^{-4}	2.13×10^{1}	3.48×10^{-2}
s(m)	0.5	0.5	0.5	0.5	0.5	0.5
B	0.938	0.723	0.711	0.929	0.818	0.875
B_1	4.72×10^{-3}	2.47×10^{-1}	2.38×10^{-1}	1.34×10^{-2}	1.19×10^{-1}	1.20×10^{-2}
B_2	1.151	0.974	1.008	1.243	1.415	1.136
t_{50}(s)	3.72×10^{-1}	3.42×10^{2}	3.77×10^{2}	4.11×10^{-4}	5.88×10^{1}	1.49×10^{-1}
t_{95}(s)	1.61×10^{0}	1.43×10^{3}	1.63×10^{3}	1.78×10^{-3}	2.54×10^{2}	6.36×10^{-1}

[a]From Rice and Cleary (1976).
[b]From Witherspoon et al. (1980).
[c]From Touloukian et al. (1989).
[d]From Ryan et al. (1977).
[e]$n_2 = 0.1n_1$ for sedimentary rock.
[f]$n_2 = 10.0n_1$ for crystalline and metamorphic rock.

fusion is most likely from the fractures into the circumscribed blocks. The rate of this process may be considered as an indication of the relative importance of the dual-porosity effect. The equilibration rate is controlled by a dimensionless time given in (23a) and (23b) as $t_D = (\gamma_{11} + \gamma_{22})t/\gamma_{12}$, whereby the time to 50% and 95% equilibration of pressures are given by $t_D^{50} = 0.6931$ and $t_D^{95} = 2.9957$. In terms of the physical parameters of the system

$$t_D = \frac{K\left(\frac{\alpha_{22}}{C_1} + \frac{\alpha_{11}}{C_2} + 2\right)t}{3D_{12}(\alpha_{11}\alpha_{22} - C_1C_2)} \quad \text{(33)}$$

where, in addition to the material coefficients controlling deformation, the block permeability, k_i, and appropriate diffusion lengths are incorporated.

The behavior of representative sandstones in Table 1 are characterized by very rapid response times, reaching t^{95} in less than a second. The low-permeability matrix of the crystalline rock impedes equilibration that, dependent on permeability and fracture spacing, may extend over thousands of seconds. Characteristic responses are illustrated in Fig. 2 to illustrate the form of the response. With drainage potential proportional to fracture spacing s^2, flatter responses in time are elicited where the path length is increased.

Although parametric analyses previously were limited to sandstones and crystalline rock, the major trends in behavior are apparent. Firstly, fluid pressures developed in compliant fractures may considerably exceed those developed in the porous body. These differential pressures will only be significant, however, if matrix permeabilities are sufficiently low that the pressure differential may be sustained in time. From the time scales apparent for permeable (sandstones) and impermeable (crystalline) materials, it appears that dual-porosity effects may be of little consequence in permeable media, but drainage time scales are sufficient in low-permeability rocks to cause noticeable effect. Where permeability controls behavior, clays and low-permeability silts may be subject to dual-porosity effects.

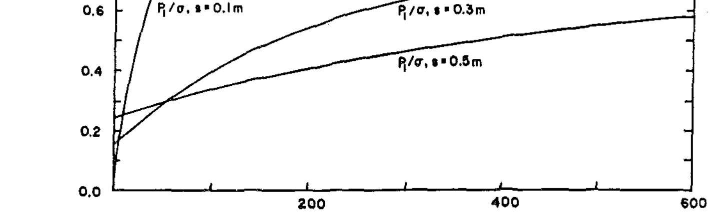


FIG. 2. Undrained Pore Pressure Response for Fractured Sandstone and Fractured Marble

GLOBAL BEHAVIOR

It is possible to solve a broader range of problems only if the previously developed constitutive behaviors are globally framed to represent spatial interaction. This requires that the constitutive, load-deformation relationship of (7) is substituted into an equilibrium statement and that the flow continuity (11) and (12) have an appropriate constitutive equation applied. The constitutive relationship, in this instance, is supplied by Darcy's law (Bear 1972).

For the general representation of heterogeneous, mixed–initial value problems, the finite element method is chosen. The equilibrium statement in its most general form is given as

$$\int_v \mathbf{B}^T \partial\boldsymbol{\sigma}\, dV - \partial\mathbf{f} = 0 \qquad (34)$$

where $\mathbf{B}$ = the strain-displacement matrix; $\partial\mathbf{f}$ = an incremental vector of applied boundary tractions, and the integration is completed over the volume of the domain (dV). Defining all quantities in terms of nodal variables

$$\partial\boldsymbol{\varepsilon} = \mathbf{B}\partial\mathbf{u} \qquad (35a)$$

$$\partial p_1 = \mathbf{N}\partial\mathbf{p}_1 \qquad (35b)$$

$$\partial p_2 = \mathbf{N}\partial\mathbf{p}_2 \qquad (35c)$$

where $\partial\mathbf{u}$ = a vector of incremental nodal displacements; $\partial\mathbf{p}$ = a vector of nodal pressures; and $\mathbf{N}$ = a vector of shape functions interpolating fluid pressures. Substituting (7) and (35) into (34), dividing by ∂t, and rearranging gives the incremental equation in finite element format as

$$\int_v \mathbf{B}^T\mathbf{D}_{12}\mathbf{B}\, dV\, \dot{\mathbf{u}} + \int_v \mathbf{B}^T\mathbf{D}_{12}\mathbf{C}_1\mathbf{m}\mathbf{N}\, dV\, \dot{\mathbf{p}}_1$$

$$+ \int_v \mathbf{B}^T\mathbf{D}_{12}\mathbf{C}_2\mathbf{m}\mathbf{N}\, dV\, \dot{\mathbf{p}}_2 = \dot{f} \qquad (36)$$

where a superscript dot identifies time derivative.

Darcy's law must be added to the continuity requirements already imposed in (11) and (12) and may be simply stated as

$$\mathbf{v}_1 = \frac{-k_1}{\mu} \nabla(p_1 + \gamma Z) \qquad (37)$$

where γ = the unit weight of the fluid; μ = the dynamic viscosity; k_1 = the porous medium permeability with the fracture permeability set to zero; and Z = the elevation of the control volume. Substituting (37) into (11) and applying the Galerkin principle results in

$$-\frac{1}{\mu}\int_v \mathbf{A}^T\mathbf{k}_1\mathbf{A}\, dV\, \mathbf{p}_1 + \int_v \mathbf{N}^T\mathbf{m}^T\mathbf{C}_1\mathbf{D}_{12}\mathbf{B}\, dV\, \dot{\mathbf{u}} - \alpha_1 \int_v \mathbf{N}^T\mathbf{N}\, dV\, \dot{\mathbf{p}}_1$$

$$- K\int_v \mathbf{N}^T\mathbf{N}\, dV(\mathbf{p}_1 - \mathbf{p}_2) = \frac{\gamma}{\mu}\int_v \mathbf{A}^T\mathbf{k}_1\mathbf{A}\, dV\, \mathbf{Z} \qquad (38)$$

for the porous phase and for the fractured medium

$$-\frac{1}{\mu}\int_v \mathbf{A}^T\mathbf{k}_2\mathbf{A}\;dV\;\mathbf{p}_2 + \int_v \mathbf{N}^T\mathbf{m}^T\mathbf{C}_2\mathbf{D}_{12}\mathbf{B}\;dV\;\dot{\mathbf{u}} - \alpha_2\int_v \mathbf{N}^T\mathbf{N}\;dV\;\dot{\mathbf{p}}_2$$

$$+ K\int_v \mathbf{N}^T\mathbf{N}\;dV(\mathbf{p}_1 - \mathbf{p}_2) = \frac{\gamma}{\mu}\int_v \mathbf{A}^T\mathbf{k}_2\mathbf{A}\;dV\;\mathbf{Z} \quad \text{(39)}$$

where $\mathbf{k}_2$ = a matrix of fracture permeabilities.

Eqs. (36), (38), and (39) may be written at any time level (say, $t + \Delta t$) and are most conveniently represented in matrix form as

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & (-\mathbf{K}_1 - \mathbf{S}_3) & \mathbf{S}_3 \\ 0 & \mathbf{S}_3 & (-\mathbf{K}_2 - \mathbf{S}_3) \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}^{t+\Delta t}$$

$$+ \begin{bmatrix} \mathbf{F} & \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{E}_1 & -\mathbf{S}_1 & 0 \\ \mathbf{E}_2 & 0 & -\mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \end{bmatrix}^{t+\Delta t} = \begin{bmatrix} \dot{\mathbf{f}} \\ \mathbf{K}_1\gamma\mathbf{Z} \\ \mathbf{K}_2\gamma\mathbf{Z} \end{bmatrix}^{t+\Delta t} \quad \text{(40)}$$

where all submatrices are defined in Appendix I.

These coupled equations remain symmetric, but they must be rearranged prior to solution in time. All terms on the right-hand side are known. The matrix relation may be integrated in time by using any convenient representation of the time derivatives. Using a fully implicit scheme, such that

$$\dot{\mathbf{u}}^{t+\Delta t} = \frac{1}{\Delta t^{t+\Delta t}}(\mathbf{u}^{t+\Delta t} - \mathbf{u}^t) \quad \text{(41a)}$$

$$\dot{\mathbf{p}}_1^{t+\Delta t} = \frac{1}{\Delta t^{t+\Delta t}}(\mathbf{p}_1^{t+\Delta t} - \mathbf{p}_1^t) \quad \text{(41b)}$$

$$\dot{\mathbf{p}}_2^{t+\Delta t} = \frac{1}{\Delta t^{t+\Delta t}}(\mathbf{p}^{t+\Delta t} - \mathbf{p}_2^t) \quad \text{(41c)}$$

and substituting (41) into (40) gives

$$\frac{1}{\Delta t^{t+\Delta t}} \begin{bmatrix} \mathbf{F} & \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{E}_1 & (-\mathbf{K}_1\Delta t^{t+\Delta t} - \mathbf{S}_1 - \Delta t\mathbf{S}_3) & \Delta t\mathbf{S}_3 \\ \mathbf{E}_2 & \Delta t\mathbf{S}_3 & (-\mathbf{K}_2\Delta t^{t+\Delta t} - \mathbf{S}_2 - \Delta t\mathbf{S}_3) \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}^{t+\Delta t} = \frac{1}{\Delta t^{t+\Delta t}} \begin{bmatrix} \mathbf{F} & \mathbf{G} & \mathbf{G}_2 \\ \mathbf{E}_1 & -\mathbf{S}_1 & 0 \\ \mathbf{E}_2 & 0 & -\mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}^{t} + \begin{bmatrix} f \\ \mathbf{K}_1\gamma\mathbf{Z} \\ \mathbf{K}_2\gamma\mathbf{Z} \end{bmatrix}^{t+\Delta t} \quad \text{(42)}$$

where the matrix comprising the left-hand side is time invariant if a constant time-step magnitude is chosen. Initial undrained conditions are recovered if $\mathbf{K}_1 = \mathbf{K}_2 = 0$ and $K = 0$ allowing (42) to be solved for $\mathbf{u}$, $\mathbf{p}_1$ and $\mathbf{p}_2$.

A two-dimensional formulation is chosen to illustrate the potential of the dual-porosity formulation. Suitable shape functions must be chosen to represent the quadratic displacement field and bilinear fluid pressure variation present within individual elements. A four-node quadrilateral isoparametric element with quadratic-incompatible modes is used. The incompatible modes are condensed out at the element level.

The broad range of material parameters that influence the pressure generation and dissipation response makes meaningful representation of transient behavior difficult. The versatility of the proposed technique is illustrated for a one-dimensional column for the parameters reported in Table 2. The fluid pressure response with depth is illustrated in Fig. 3 for a fracture spacing, s, of 0.1 m. Fluid pressures generated within the fractures are considerably greater than the matrix pore pressures. With time, the fracture pressures dissipate into the porous blocks until an equilibrium is reached. In this example, following the establishment of an equilibrium pressure distribution, the consolidation process will continue by drainage from the top of the layer. The surface settlement associated with the expulsion of fluid from the dual-porosity system is illustrated in Fig. 4. Fracture spacings of 0.025, 0.05, and 0.1 m are used to demonstrate the differing responses. Where spacing is decreased, the dissipation process is accelerated. The dual-porosity response is contrasted with the behavior of a single-porosity system, where the different form of settlement-versus-time behavior is apparent.

A two-dimensional geometry, represented in Fig. 5, also may be used to illustrate the essential differences between single-porosity and dual-porosity effects. The response to a load of finite extent is illustrated in Fig. 6. As fracture spacing is decreased, the magnitude of initial normalized displacement increases, reflecting the dominant influence of fracture stiffnesses on the deformation behavior. The small block dimensions present within the

TABLE 2. Example Coefficients

Parameter (1)	Definition (2)	Magnitude (3)	Units (4)
E	modulus of elasticity	1.0	MN/m^2
ν	Poisson's ratio	0.15	
k_n	fissure stiffness	0.1	$MN/m^2/m$
K_f	fluid bulk modulus	0.1	MN/m^2
n_1	matrix porosity	0.1	
n_2	fissure porosity	0.05	
k_1/μ	matrix permeability	0.01×10^{-3}	$m^4/(MN \cdot s)$
k_2/μ	fissure permeability	0.1	$m^4/(MN \cdot s)$
s	fissure spacing	0.025, 0.05, 0.1	m

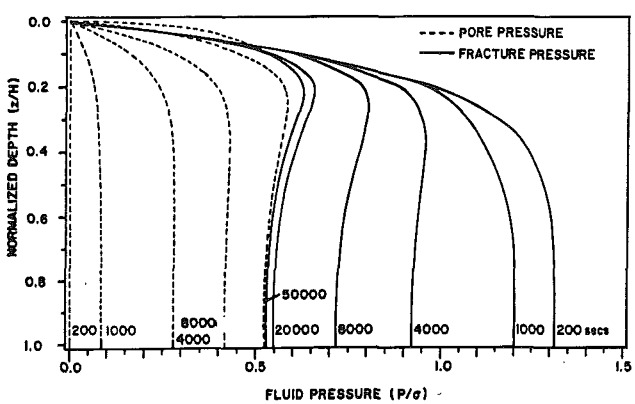


FIG. 3. Pore Pressure Equilibration Response for One-Dimensional Column Loaded Axially

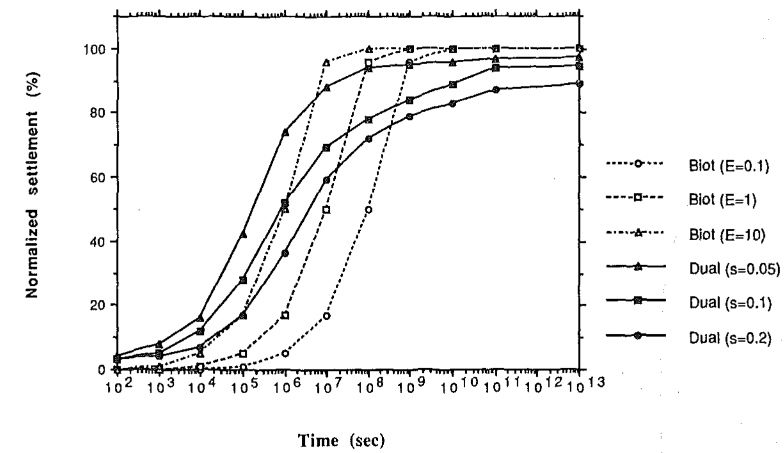


FIG. 4. Surface Displacement Response for One-Dimensional Column Loaded Axially

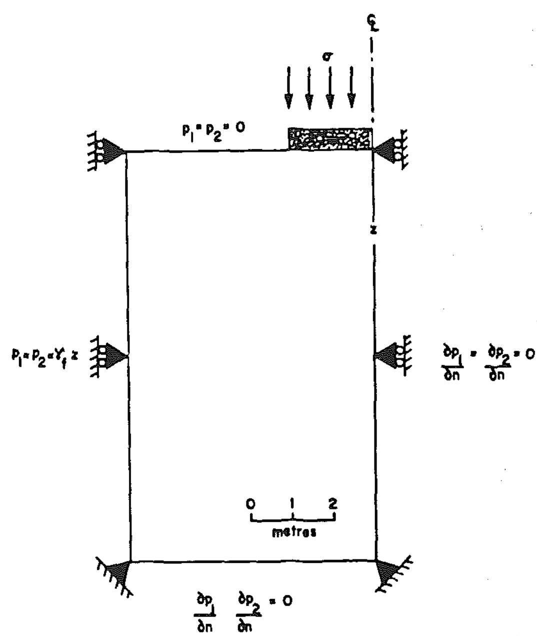


FIG. 5. Two-Dimensional Geometry

closely fractured medium result in an acceleration of the consolidation process. As bulk permeability magnitude is correspondingly increased, the dual-porosity behavior exhibits a characteristic acceleration in the early time response and a deceleration in the late time response over the equivalent single-porosity behavior. The former characteristic results from the rapid equilibrating of fluid pressures within the fractures where dissipation increases the magnitude of effective stresses. The high compressibility of the fractures correspondingly yields a magnified response. At later times, the dual porosity exhibits a more sluggish response over the equivalent single-porosity behavior as a result of delayed changes in effective stresses within the porous blocks. This behavior is slightly magnified in time as block dimension is increased, resulting in a longer dissipation tail, as evident in Fig. 6.

CONCLUSIONS

A constitutive model for the coupled flow-deformation response of dual-porosity media has been presented. The formulation is stated in component terms, where behavior is controlled by the mechanical and hydraulic response of the individual porous and fractured phases. Representation in this form allows the potential importance of dual-porosity effects to be evaluated in controlling the coupled flow-deformation behavior of fractured geologic media.

Pore pressure coefficients B_1 and B_2 may be defined for the dual-porosity response, where the magnitudes are not necessarily confined to the range between zero and unity. For realistic material parameters, the magnitude of the pore pressure coefficient for the fractures is commonly greater than unity, and for the porous medium, less than unity. With the generation of this pressure differential, fluid transfer between the porous medium and the fracture will attempt to equilibrate the pressures. Equilibration times are controlled by the compliances of the porous medium and the fractures, together with porosities and the block-fracture fluid-transfer coefficient K. This latter component is a function of block size and block permeability. Where undrained loading is prescribed, the times to 50% and 95% equilibration may be determined explicitly.

Where general boundary conditions are applied to the dual-porosity system, a numerical solution method must be employed, Representation as a finite element system is particularly appropriate, allowing spatial variability in parameters and general boundary conditions to be readily accommodated.

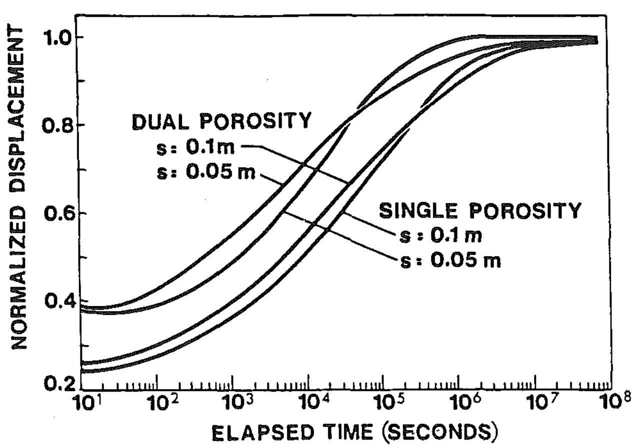


FIG. 6. Surface Displacement Response at Centerline for Single-Porosity and Dual-Porosity Systems (parameters Given in Table 2)

The procedure may be applied to a variety of engineering situations where the accurate determination of parameters associated with dual-porosity systems is important. These include the performance of fractured clay tills used as barriers to contaminated ground water and waste leachate and the behavior of fractured rocks deforming above mined underground openings. In either instance, the behavior of the dual-porosity medium is distinctively different from that of a single-porosity continuum, and it is important to recognize this differentiation if serviceable designs are to be commissioned. For example, large initial deformations are apparent in the dual-porosity medium under undrained loading that result in considerably different transient response to anything that can be explained using single-porosity representation. Thus, use of an incorrect phenomenological model in describing field data would erroneously predict the performance of the prototype. For this reason, it is imperative that a correct and adequate model is initially selected.

Acknowledgments

Support provided by the Standard Oil Center for Scientific Excellence and the Waterloo Centre for Ground Research is most gratefully acknowledged.

Appendix I. Submatrices

$$\mathbf{E}_1 = \int_v \mathbf{N}^T\mathbf{m}^T\mathbf{C}_1\mathbf{D}_{12}\mathbf{B}\, dV \quad (43)$$

$$\mathbf{E}_2 = \int_v \mathbf{N}^T\mathbf{m}^T\mathbf{C}_2\mathbf{D}_{12}\mathbf{B}\, dV \quad (44)$$

$$\mathbf{F} = \int_v \mathbf{B}^T\mathbf{D}_{12}\mathbf{B}\, dV \quad (45)$$

$$\mathbf{G}_1 = \int_v \mathbf{B}^T\mathbf{D}_{12}\mathbf{C}_1\mathbf{m}\mathbf{N}\, dV \quad (46)$$

$$\mathbf{G}_2 = \int_v \mathbf{B}^T\mathbf{D}_{12}\mathbf{C}_2\mathbf{m}\mathbf{N}\, dV \quad (47)$$

$$\mathbf{H} = \int_v \mathbf{N}^T\mathbf{N}\, dV \quad (48)$$

$$\mathbf{K}_1 = \frac{1}{\mu}\int_v \mathbf{A}^T\mathbf{k}_1\mathbf{A}\, dV \quad (49)$$

$$\mathbf{K}_2 = \frac{1}{\mu}\int_v \mathbf{A}^T\mathbf{k}_2\mathbf{A}\, dV \quad (50)$$

$$\mathbf{S}_1 = \alpha_1\mathbf{H} \quad (51)$$

$$\mathbf{S}_2 = \alpha_2\mathbf{H} \quad (52)$$

$$\mathbf{S}_3 = K\mathbf{H} \quad (53)$$

APPENDIX II. REFERENCES

Aifantis, E. C. (1977). "Introducing a multi-porous medium." *Developments in mechanics*, 8, 209–211.

Aifantis, E. C. (1980). "On the problem of diffusion in solids." *Acta Mechanica*, 37, 265–296.

Bear, J. (1971). *Dynamics of fluids in porous media*. American Elsevier, New York, N.Y.

Bibby, R. (1981). "Mass transport of solutes in dual porosity media." *Water Resour. Res.*, 17, 1075–1081.

Biot, M. A. (1941). "General theory of three-dimensional consolidation." *J. Appl. Phys.* 12, 151–164.

Elsworth, D. (1989). "Thermal permeability enhancement of blocky rocks: one dimensional flows." *Int. J. Rock Mech. Min. Sci. Geomech. Abstr.*, 26(3/4), 329–339.

Goodman, R. E. (1974). "The mechanical properties of joints." *Proc. of the Third Congress of Int. Soc. for Rock Mech.*, 127–140.

Huyakorn, P. S., Lester, B. H., and Faust, C. R. (1983). "Finite element techniques for modeling groundwater flow in fractured aquifers." *Water Resour. Res.*, 19, 1019–1035.

Iwai, K. (1976). "Fundamental studies of fluid flow through a single fracture," thesis presented to the University of California, at Berkeley, California, in partial fulfillment of the requirements of the degree of Doctor of Philosophy.

Kazemi, H. (1969). "Pressure transient analysis of naturally fractured reservoirs with uniform fracture distribution." *Soc. Pet. Engrg. J.*, 9, 451–462.

Kazemi, H., Merrill, L. S., Jr., Porterfield, K. L., and Zeman, P. R. (1976). "Numerical simulation of water-oil flow in naturally fractured reservoirs." *Soc. Pet. Engrg. J.*, 16, 317–326.

Kazemi, H., and Merrill, L. S. Jr. (1979). "Numerical simulation of water imbibition in fractured cores." *Soc. Pet. Engrg. J.*, 19, 175–182.

Khaled, M. Y., Beskos, D. E., and Aifantis, E. C. (1984). "On the theory of consolidation with double porosity—III. A finite element formulation." *Int. J. Numer. Anal. Meth.* Geomech., 8, 101–123.

Mandel, J. (1953). "Consolidation des sols (étude mathematique)." *Geotechnique*, 3, 287–299.

Nur, A., and Byerlee, J. D. (1971). "An exact effective stress law for elastic deformation of rock with fluids." *J. Geophys. Res.*, 76(26), 6414–6419.

Odeh, A. S. (1965). "Unsteady-state behavior of naturally fractured reservoirs." *Soc. Pet. Engrg. J.*, 5, 60–66.

Pruess, K., and Narasimhan, T. N. (1985). "A practical method for modelling fluid and heat flow in fractured porous media." *Soc. Pet. Engrg. J.*, 25, 14–26.

Rice, J. R., and Cleary, M. P. (1976). "Some basic stress diffusion solutions for fluid saturated elastic media with compressible constituents." *Rev. Geophys. Space Phys.*, 14(2), 227–241.

Ryan, T. M., Farmer, I., and Kimbrell, A. F. (1977). "Laboratory determination of fracture permeability." *Proc. of the 18th U.S. Symp. on Rock Mechanics*,

Skempton, A. W. (1960). "Effective stress in soils, concrete and rock." *Proc. of the Symp. on Pore Pressure and Suction in Soils*, Butterworths, London, U.K., 1.

Terzaghi, K. (1923). "Die Berechnung der Durchasigkeitsziffer des Tones aus dem Verlauf der hydrodynamishen Spannungserscheinungen, Sitzungsber." *Acad. Wiss. Wien Math Naturwiss.* Kl. Alot. 2A, 132, 105 (in German).

Terzaghi, K. (1943). *Theoretical soil mechanics*, John Wiley & Sons, New York, N.Y.

Thomas, L. K., Dixon, N. T., and Pierson, G. R. (1983). "Fractured reservoir simulation." *Soc. Pet. Engrg. J.*, 23, 42–54.

Touloukian, Y. S., Judd, W. R., and Roy, R. F. (1989). *Physical properties of rocks and minerals*. 11(2), McGraw-Hill, New York, N.Y.

Warren, J. E., and Root, P. J. (1963). "The behavior of naturally fractured reservoirs." *Soc. Pet. Engrg. J.*, 3, 245–255.

Wilson, R. K., and Aifantis, E. C. (1982). "On the theory of consolidation with double porosity." *Int. J. Engrg. Sci.*, 20(9), 1009–1035.
Witherspoon, P. A., Wang, J. S. Y., Iwai, K., and Gale, J. E. (1980). "Validity of cubic law for fluid flow in a deformable structure." *Water Resour. Res.*, 16(16), 1016–1024.
Yamamoto, R. H., Padgett, J. B., Ford, W. T., and Boubeguira, A. (1971). "Compositional reservoir simulator for fissured systems—the single block model." *Soc. Pet. Engrg. J.*, 11, 113–128.

APPENDIX III. NOTATION

The following symbols are used in this paper:

$\mathbf{A}$ = derivatives of shape function matrix $\mathbf{N}$;
$\mathbf{B}$ = strain displacement matrix;
B_i = Skempton's fluid pressure coefficients in true dual-porosity system;
B = Skempton's fluid pressure coefficients in pseudodual-porosity system;
$\mathbf{C}_i$ = compliance matrices;
$\mathbf{D}_i$ = elasticity matrices;
$\mathbf{D}_{12}$ = elasticity matrix $(\mathbf{C}_1 + \mathbf{C}_2)^{-1}$;
E = Young's modulus;
$\mathbf{f}$ = vector of boundary tractions;
K = fluid transfer coefficient;
k_i = permeabilities;
k_n = joint normal stiffness;
K_f = fluid bulk modulus;
K_s = solid grain bulk modulus;
$\mathbf{m}$ = one dimensional vector; for three-dimensional problem, m^T = (1 1 1 0 0 0); for two-dimensional problem, m^T = (1 1 0);
$\mathbf{m}_i$ = modified intergranular stress relationship;
$\mathbf{m}_n$ = $(\mathbf{m} - 1/3Ks\mathbf{D}_n\mathbf{m})$;
n_i = porosities;
p_i = fluid pressure;
s = fracture spacing;
t = time;
t_D = dimensionless time;
$\mathbf{u}$ = displacement vector;
$\mathbf{v}_i$ = flow velocity vector;
Z = elevation of control volume;
∂ = partial differential operator;
α_i = hydrostatic compressibility;
γ = unit weight of fluid;
∂ = partial differential operator;
$\boldsymbol{\varepsilon}$ = strain vector; in three-dimensional Cartesian coordinates, $\boldsymbol{\varepsilon}^T = (\varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz}\gamma_{xy}\gamma_{xz}\gamma_{yz})$;
μ = dynamic viscosity;
ν = Poisson ratio; and
$\boldsymbol{\sigma}$ = stress vector; in three-dimensional Cartesian coordinates, $\sigma^T = \sigma_{xx}\sigma_{yy}\sigma_{zz}\sigma_{xy}\sigma_{xz}\sigma_{yz})$.

Subscripts

i = 1, 2, denoting pore phase and fracture phase, respectively.

4:4 Mechanical deformation - 1D and 2D Elements [5:1][5:2]

http://youtu.be/WQbG588gOg4

[5:1] Solid Mechanics

Principle of virtual work

1D element

2D element

PROCESS COUPLINGS [T-H-M-C]

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{p} \\ \underline{T} \\ \underline{c} \end{Bmatrix} + \begin{bmatrix} S_{11} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{Bmatrix} \underline{\dot{u}} \\ \underline{\dot{p}} \\ \underline{\dot{T}} \\ \underline{\dot{c}} \end{Bmatrix} = \begin{Bmatrix} \underline{\dot{f}} + \ldots \\ \underline{q}_F + \ldots \\ \underline{q}_T + \ldots \\ \underline{q}_M + \ldots \end{Bmatrix}$$

FINAL EQUATION $\quad \underline{f} = \underline{K}\,\underline{u} \quad$ or $\quad \underline{\dot{f}} = \underline{K}\,\underline{\dot{u}}$

$$K = \int_V \underline{a}^T \underline{D}\, \underline{a}\, dV$$

$$\begin{cases} \underline{\varepsilon} = \underline{a}\,\underline{u} \quad \text{or} \quad \underline{\dot{\varepsilon}} = \underline{a}\,\underline{\dot{U}} \\ u = \underline{b}\,\underline{U} \\ \underline{\sigma} = \underline{D}\,\underline{\varepsilon} \end{cases}$$

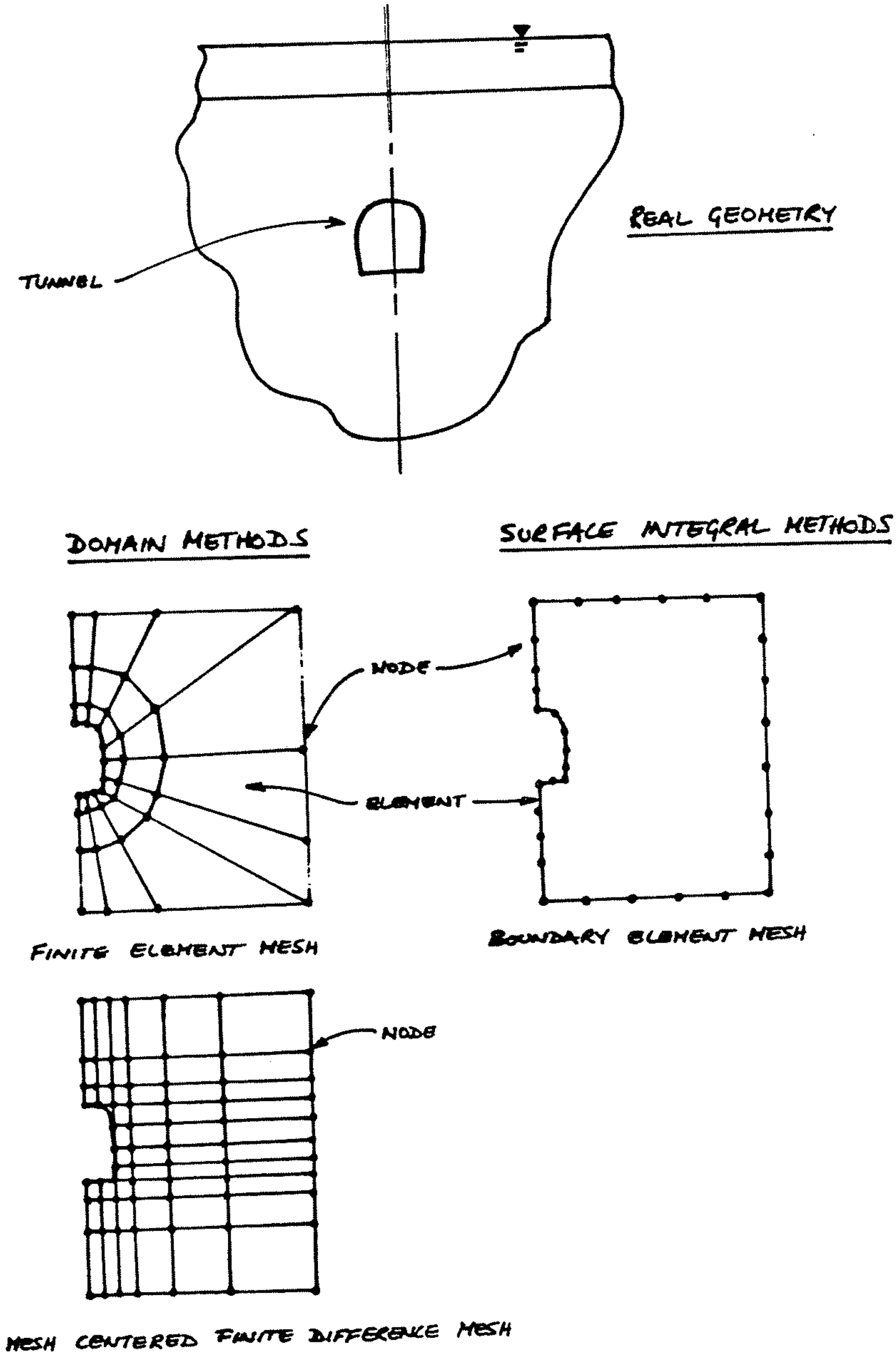


Figure 1.1 Domain and Integral Representations of a Flow Problem

SYSTEM TYPES

SOLID MECHANICS

- Conservation of momentum: (Equilibrium), $VW_I = VW_E$
- Continuity (Compatibility): $\underline{\varepsilon} = \underline{a}\,\underline{u}$
- Constitutive relation: $\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$
- Initial Conditions
- Boundary Conditions

FLOW SYSTEM

- Conservation of mass: $\underline{\nabla}^T \underline{q} = 0$
- Continuity: $\underline{h}_, = \underline{a}\,\underline{h}$
- Constitutive reln. $\underline{v} = \underline{D}\,\underline{h}_,$
- ICs
- BCs

TRANSPORT

- Conservation of mass $\underline{\nabla}^T \underline{q} = 0$
- Continuity: $\underline{c}_, = \underline{a}\,\underline{c}$
- Constitutive:
  - diffusion - $\underline{v}_1 = \underline{D}\,\underline{c}_,$
  - advective - $\underline{v}_2 = \underline{A}\,\underline{c}$
- ICs
- BCs

— SOLVE SYSTEM EQUATIONS —

BASIC EQUATIONS – COMMONALITY IN SOLUTION

FLOW $\underline{q} = \underline{K}\,\underline{h}$

$$\underline{K} = \int_V \underline{a}^T \underline{D}\,\underline{a}\,dV$$
$$\underline{h}_, = \underline{a}\,\underline{h}$$
$$\underline{v} = \underline{D}\,\underline{h}_,$$

TRANSPORT $\underline{q} = \underline{K}\,\underline{c}$

$$\underline{K} = \underline{K}_1 + \underline{K}_2$$
$$\underline{K}_1 = \int_V \underline{a}^T \underline{D}\,\underline{a}\,dV$$
$$\underline{K}_2 = \int_V \underline{b}^T \underline{v}\,\underline{a}\,dV$$

SOLID MECHANICS $\underline{f} = \underline{K}\,\underline{u}$

$$\underline{K} = \int_V \underline{a}^T \underline{D}\,\underline{a}\,dV$$
$$\underline{\varepsilon} = \underline{a}\,\underline{u}$$
$$\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$$

* ALL EQUATIONS REDUCE TO A SET OF SIMULTANEOUS ALGEBRAIC EQUATIONS TO BE SOLVED FOR THE DEPENDENT VARIABLES ONCE BOUNDARY CONDITIONS ARE APPLIED.

* FEM CODES ARE STRUCTURED TO ALLOW:

- VARIABLE ELEMENT TYPES
- VARIABLE DIMENSIONALITY 1-D → 3-D
- SYMMETRIC/NONSYMMETRIC MATRICES
- ITERATIVE SOLUTION

SOLID MECHANICS – PRINCIPLE OF VIRTUAL WORK

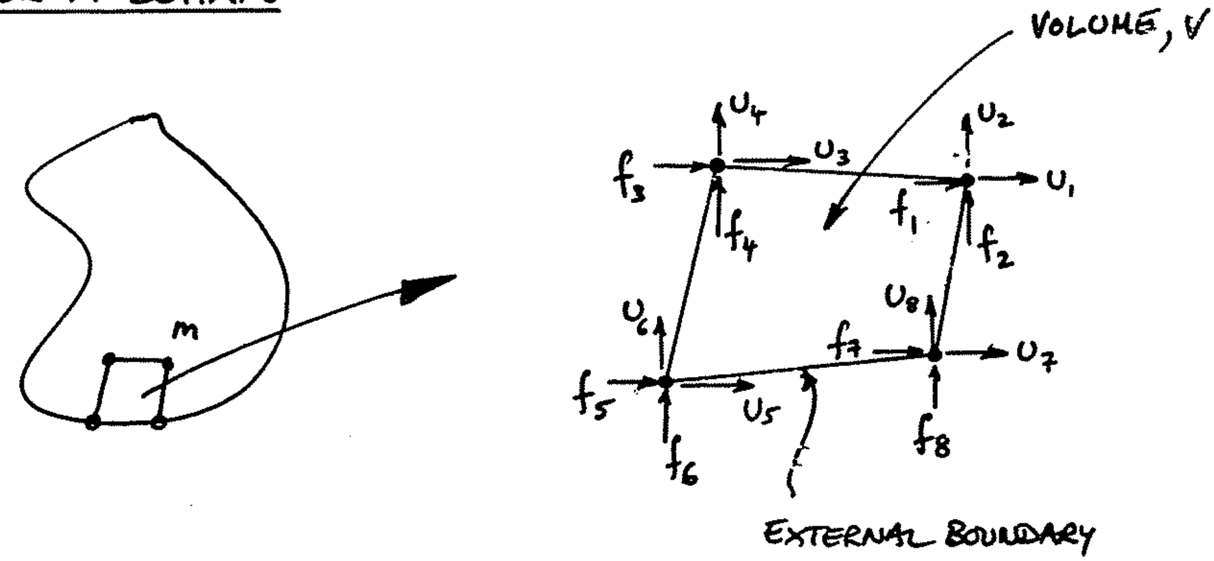
- Turner, Martin, Clough & Topp (1956) ; Argyris (1955)

PRINCIPLE OF VIRTUAL WORK (VW)

Definition: An elastic body is in equilibrium, if, for any set of virtual displacements, $\bar{\underline{u}}$, the virtual work of the external forces is equal to the virtual strain energy of the system (i.e. internal work)

$$\boxed{VW_E = VW_I} \qquad (1)$$

CONSIDER A DOMAIN



Isolate element, m

$$\boxed{VW_E^{(m)} = VW_I^{(m)}} \qquad (2)$$

EXTERNAL VIRTUAL WORK, VW_E

$$VW_E = \text{Boundary forces} \times \text{Boundary displacements (virtual)}$$

For element with n degrees of freedom (example has $n = 8$):

$$\underline{u}^T = [u_1, u_2 \;\cdots\; u_n] \quad ; \quad \underline{f}^T = [f_1, f_2 \;\cdots\cdots\; f_n]$$

Sign convention: bar subscript $\equiv$ vector   e.g. $\underline{u}$

bar superscript $\equiv$ virtual   e.g. $\bar{\underline{u}}$

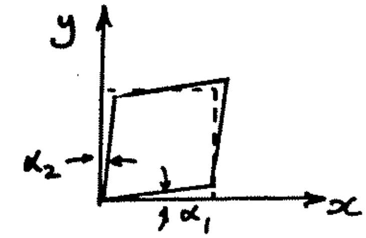
then $$\boxed{VW_E = \sum_{i=1}^{n} \bar{u}_i f_i = \bar{\underline{u}}^T \underline{f}} \qquad (1)$$

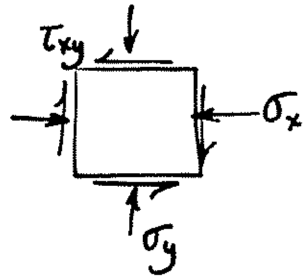
INTERNAL VIRTUAL WORK, VW_I

VW_I = Body strain resulting from virtual displ. $\times$ Stress change due to boundary conditions

For 2-D system: $$\underline{\varepsilon}^T = [\varepsilon_x ; \varepsilon_y ; \gamma_{xy}] ; \quad \underline{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$\gamma_{xy} = (\alpha_1 + \alpha_2)$





FOUR REQUIREMENTS TO SOLVE THE ELASTIC BOUNDARY VALUE PROBLEM

Equilibrium: Satisfied if $VW_E = VW_I$ (2)

Constitutive Law: $\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$ (3)

$\underline{D}$ = Elasticity matrix (3×3)

Strain-displacement (compatability): $\underline{\varepsilon} = \underline{a}\,\underline{u}$ (4)

Boundary conditions: supplied

Definition of $VW_I = \int_{Vol} \underline{\bar{\varepsilon}}^T \underline{\sigma}\, dV$ (5)

Substituting (4) into (3) $\underline{\sigma} = \underline{D}\,\underline{a}\,\underline{u}$ (6)

Substituting (6) into (5) $VW_I = \int_V (\underline{a}\,\underline{\bar{u}})^T \underline{D}\,\underline{a}\,\underline{u}\, dV$ (7)

Note that $(\underline{x}\,\underline{y})^T = \underline{y}^T \underline{x}^T$ then; (7) becomes

$$\boxed{VW_I = \underline{\bar{u}}^T \int_V \underline{a}^T \underline{D}\,\underline{a}\, dV\, \underline{u}} \quad (8)$$

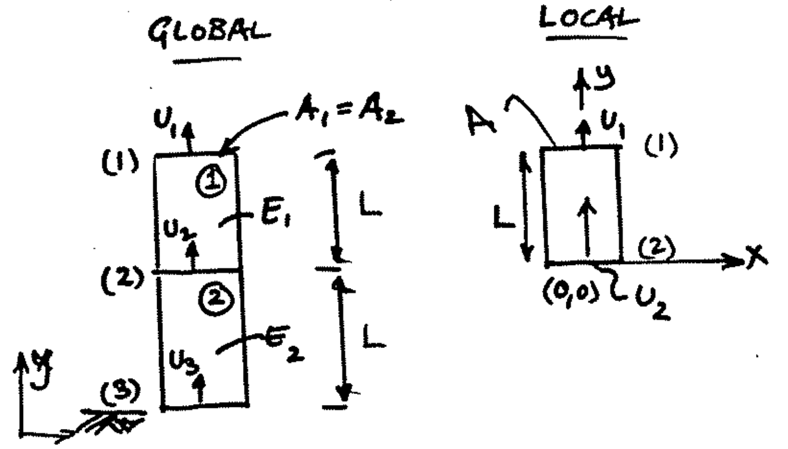
Recall that: $$\boxed{VW_E = \underline{\bar{u}}^T \underline{f}} \quad (9)$$

EQUATE

Equating (8) & (9) $\cancel{\underline{\bar{u}}^T}\underline{f} = \cancel{\underline{\bar{u}}^T}\underbrace{\int_V \underline{a}^T \underline{D}\,\underline{a}\, dV}_{\underline{K}}\, \underline{u}$ (10)

$$\boxed{\underline{f} = \underline{K}^{(m)}\underline{u}}$$

EXAMPLE 1-D PROBLEM



For element ①;

$$\underline{f} = \underline{K}\,\underline{u} \qquad (1)$$

$$\underline{K} = \int_V \underline{a}^T \underline{D}\, \underline{a}\, dV \quad (2) \quad \text{with} \quad \begin{aligned} \underline{\sigma} &= \underline{D}\,\underline{\varepsilon} && (3) \\ \underline{\varepsilon} &= \underline{a}\,\underline{u} && (4) \end{aligned}$$

D matrix $\qquad \underline{D} = [E_1] \qquad$ since $\quad E = \dfrac{\sigma}{\varepsilon} \qquad (5)$

---

a matrix $\qquad \underline{\varepsilon} = \dfrac{\partial U_y}{\partial y} = \dfrac{\partial}{\partial y}(U_y) \qquad (6)$

assume $\quad u_y = U_2 + (U_1 - U_2)\dfrac{y}{L} \qquad (7)$

Then $\quad u_y = U_1$ @ node ① when $y = L$

$\qquad\quad u_y = U_2$ @ node ② when $y = 0$ $\qquad$ QED.

Substituting (7) into (6) then:

$$\underline{\varepsilon} = \frac{\partial}{\partial y}(U_y) = (U_1 - U_2)\frac{1}{L} = \underbrace{\frac{1}{L}\,[1 \; ; \; -1]}_{\underline{a}} \underbrace{\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}}_{\underline{u}} \qquad (8)$$

Substituting (5) and (8) into (2)

$$\underline{K} = \int_V \underline{a}^T \underline{D}\,\underline{a}\, dV = A \int_0^L \frac{1}{L}\begin{bmatrix} 1 \\ -1 \end{bmatrix} [E_1]\, \frac{1}{L}[1 \; ; \; -1]\; dy \qquad (9)$$

Rearranging (9) : $$K = A\int_0^L \frac{E}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dy \qquad (10)$$

Taking individual terms from (10), then $\frac{AE}{L^2}\int_0^L 1\, dy = \frac{AE}{L}$

and resubstituting into (10) then:

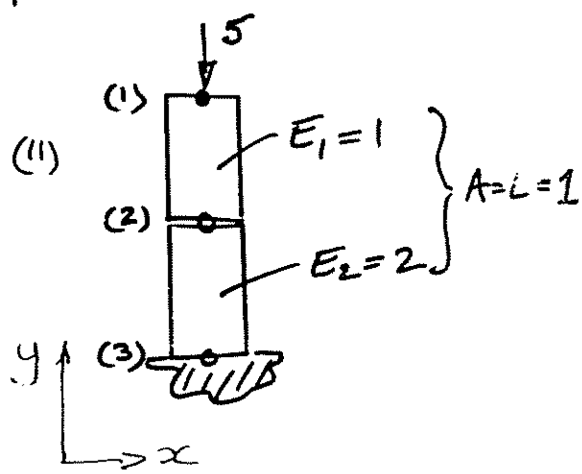
$$\boxed{\underline{K}^{(1)} = \frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \quad (1)$$

and similarly for (2)

$$\boxed{\underline{K}^{(2)} = \frac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \quad (2)$$

MOVING TO THE GLOBAL LEVEL $\sum_1^m VW_E^{(m)} = \sum_1^n VW_{\phi_I}^{(m)}$

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \qquad 2 \qquad & \quad 3 \quad \end{array} \\ \begin{bmatrix} \frac{AE_1}{L_1} & -\frac{AE_1}{L_1} & 0 \\ -\frac{AE_1}{L_1} & \left(\frac{AE_1}{L_1} + \frac{AE_2}{L_2}\right) & -\frac{AE_2}{L_2} \\ 0 & -\frac{AE_2}{L_2} & \frac{AE_2}{L_2} \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad (11)$$



Then the system of equations becomes:

$$\begin{Bmatrix} -5 \\ f_2 \\ \underline{f_3} \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & (1+2) & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ 0 \end{Bmatrix} \quad (12)$$

u_1 ; u_2 ; f_3 are unknowns

$f_2 = 0$ since internal node with no <u>prescribed</u> force.

Rearrange (12) for boundary conditions:

$$\left\{\begin{matrix} -5-(0)(0) \\ 0-(-2)(0) \\ \hdashline f_3-(2)(0) \end{matrix}\right\} = \left[\begin{matrix} 1 & -1 \\ -1 & 3 \\ \hdashline 0 & -2 \end{matrix}\right] \left\{\begin{matrix} u_1 \\ u_2 \end{matrix}\right\} \qquad (13)$$

Solve as: $\underline{u_1 = -15/2}$; $\underline{u_2 = -5/2}$

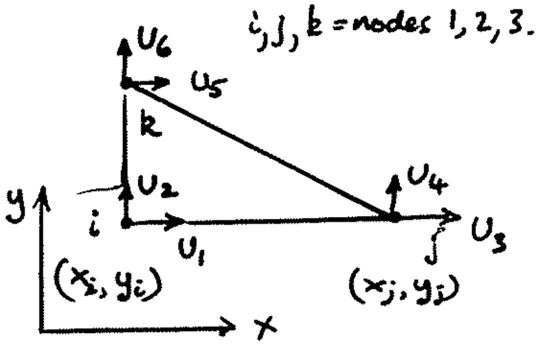
Resubstitute as: $\underline{f_3 = +5}$ ✓ QED

Alternatively: Shortening of element $(2) = u_2 - u_3 = -5/2$

$(1) = u_1 - u_2 = -10/2$ ✓

QED

THREE-NODED CONSTANT STRAIN ELEMENT



$$\underline{K} = \int_V \underline{a}^T \underline{D}\, \underline{a}\, dV$$

- Six degrees of freedom.
- Assume linear variation in displ. for both

U_x and U_y since 3 values of each are required to define each of two planar surfaces

$$U_x = a + bx + cy \quad \doteq \text{equation of a plane} \qquad (1)$$
$$U_y = d + ex + fy \qquad (2)$$

Strain components

$$\left.\begin{array}{llll} \varepsilon_{xx} = \partial U_x/\partial x & = & b & (3) \\ \varepsilon_{yy} = \partial U_y/\partial y & = & f & (4) \\ \gamma_{xy} = \partial U_x/\partial y + \partial U_y/\partial x & = & c + e & (5) \end{array}\right\} \text{hence strain constant.}$$

Require to obtain $(\underline{\varepsilon} = \underline{a}\,\underline{u})$ – The '$\underline{a}$' matrix.

Considering only displacements in the U_x direction first, we know that displacements U_1, U_3, and U_5 occur at their respective coordinates (x, y)

Substitution of coordinate values into equation (1) gives

$$U_1 = a + bx_i + cy_i \qquad (6)$$
$$U_3 = a + bx_j + cy_j \qquad (7)$$
$$U_5 = a + bx_k + cy_k \qquad (8)$$

or

$$\begin{Bmatrix} U_1 \\ U_3 \\ U_5 \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \qquad (9)$$

This may be inverted to obtain (by any convenient method).

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} (x_j y_k - x_k y_j) & (x_k y_i - x_i y_k) & (x_i y_j - x_j y_i) \\ (y_j - y_k) & (y_k - y_i) & (y_i - y_j) \\ (x_k - x_j) & (x_i - x_k) & (x_j - x_i) \end{bmatrix} \begin{Bmatrix} U_1 \\ U_3 \\ U_5 \end{Bmatrix} \quad (10)$$

$\Delta =$ the area of the triangular element.

Note permutation of subscripts – in rows $ijkijk\ldots$

Shorthand notation:

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_3 \\ U_5 \end{Bmatrix} \quad (11)$$

Resubstituting equation (11) into (1) gives $\quad U_x = a + bx + cy$

$$U_x = \frac{1}{2\Delta} \left[\overbrace{(\beta_{11} U_1 + \beta_{12} U_3 + \beta_{13} U_5)}^{a} + \overbrace{(\beta_{21} U_1 + \beta_{22} U_3 + \beta_{23} U_5)}^{b} x + \overbrace{(\beta_{31} U_1 + \beta_{32} U_3 + \beta_{33} U_5)}^{c} y \right] \quad (12)$$

Gives U_x at <u>any</u> point in element $\underline{U_x(x,y)}$ defined by:

- nodal displacements, U_1, U_3, U_5
- geometry of element.

Referring to equation (3) $\qquad \varepsilon_{xx} = \partial u_x / \partial x = b \quad (3)$

from equation (11)

$$\boxed{b = \varepsilon_{xx} = \frac{1}{2\Delta} (\beta_{21} U_1 + \beta_{22} U_3 + \beta_{23} U_5)} \quad (13)$$

A similar procedure may be completed for displacements only in the y direction, U_y. First substitute the nodal coords. into eqn (2)

$$\begin{Bmatrix} U_2 \\ U_4 \\ U_6 \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} d \\ e \\ f \end{Bmatrix} \quad (14)$$

Inverting (14) yeilds the same coefficient matrix of equation (10). Thus referring to the shorthand adopted in equation (11).

$$\begin{Bmatrix} d \\ e \\ f \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_4 \\ U_6 \end{Bmatrix} \qquad (15)$$

Substituting into equn(4) $\quad \varepsilon_{yy} = \frac{1}{2\Delta}\left(\beta_{31} U_2 + \beta_{32} U_4 + \beta_{33} U_6\right)$ (16)

and equn(5) $\quad \gamma_{xy} = \frac{1}{2\Delta}\left(\beta_{31} U_1 + \beta_{32} U_3 + \beta_{33} U_5 + \beta_{21} U_2 + \beta_{22} U_4 + \beta_{23} U_6\right)$ (17)

Writing ($\underline{\varepsilon} = \underline{a}\,\underline{U}$) in matrix form

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} \beta_{21} & 0 & \beta_{22} & 0 & \beta_{23} & 0 \\ 0 & \beta_{31} & 0 & \beta_{32} & 0 & \beta_{33} \\ \beta_{31} & \beta_{21} & \beta_{32} & \beta_{22} & \beta_{33} & \beta_{23} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} \qquad (18)$$

And the '$\underline{D}$' matrix ($\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$) for plane strain is;

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad (19)$$

Recap:

- Require to obtain stiffness matrix $\underline{K} = \int_V \underline{a}^T \underline{D}\, \underline{a}\, dV \qquad \underline{f} = \underline{K}\underline{u}$
- $\underline{D}$ is known since $f(E, \nu)$ only
- $\underline{a}$ may be obtained for the element as $f(\text{geometry})$ only.

II.5 Inversion (Adjoint Matrix)

It can be shown that

$$\mathbf{a}\,(\mathrm{adj}\,\mathbf{a}) = |\mathbf{a}|\,\mathbf{I} \tag{II.11}$$

where $|\mathbf{a}|$ is the determinant of the matrix $\mathbf{a}$ and adj $\mathbf{a}$, called the *adjoint* matrix, is the transpose of the matrix of cofactors of the determinant. Comparing (II.10) and (II.11) we see that

$$\mathbf{a}^{-1} = \frac{\mathrm{adj}\,\mathbf{a}}{|\mathbf{a}|} \tag{II.12}$$

from which it is clear that the inverse does not exist when $|\mathbf{a}|$ is zero, in which case $\mathbf{a}$ is said to be *singular*.

To illustrate the method we shall determine the inverse of the matrix

$$\mathbf{H} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \tag{II.13}$$

If we delete the pth row and qth column from the determinant of the matrix we obtain the minor H'_{pq}, e.g. deleting row 3 and column 1 we have

$$H'_{31} = \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \tag{II.14}$$

The cofactor $\bar{H}_{pq}$ is the product of the minor and $(-1)^{(p+q)}$. When the cofactors are written as a matrix and then transposed we have the adjoint matrix

$$\mathrm{adj}\,\mathbf{H} = \begin{bmatrix} \begin{vmatrix} x_j & y_j \\ x_m & y_m \end{vmatrix} & -\begin{vmatrix} x_i & y_i \\ x_m & y_m \end{vmatrix} & \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \\ -\begin{vmatrix} 1 & y_j \\ 1 & y_m \end{vmatrix} & \begin{vmatrix} 1 & y_i \\ 1 & y_m \end{vmatrix} & -\begin{vmatrix} 1 & y_i \\ 1 & y_j \end{vmatrix} \\ \begin{vmatrix} 1 & x_j \\ 1 & x_m \end{vmatrix} & -\begin{vmatrix} 1 & x_i \\ 1 & x_m \end{vmatrix} & \begin{vmatrix} 1 & x_i \\ 1 & x_j \end{vmatrix} \end{bmatrix} \tag{II.15}$$

For example H'_{31} of (II.14) is transposed to row 1 column 3. Expanding the determinants we have

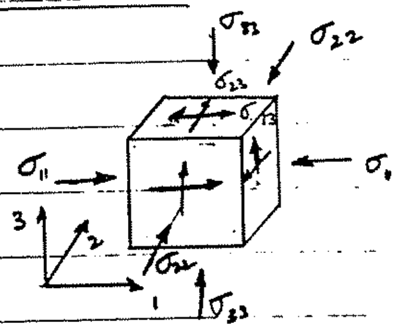
$$\mathrm{adj}\,\mathbf{H} = \begin{bmatrix} (x_j y_m - x_m y_j) & -(x_i y_m - x_m y_i) & (x_i y_j - x_j y_i) \\ -(y_m - y_j) & (y_m - y_i) & -(y_j - y_i) \\ (x_m - x_j) & -(x_m - x_i) & (x_j - x_i) \end{bmatrix} \tag{II.16}$$

The inverse is obtained by dividing adj $\mathbf{H}$ by the determinant of $\mathbf{H}$.

[5:2] Solid Mechanics

Constitutive Relations

Stress / Strain Relationships for 3-D Isotropic, Linear Elasticity



General equations

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right] \quad (1)$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right] \quad (2)$$

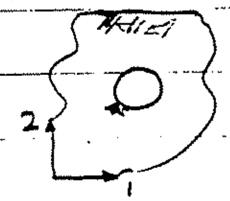
$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right] \quad (3)$$

$$\gamma_{12} = \sigma_{12}/G \; ; \; \gamma_{13} = \sigma_{13}/G \; ; \; \gamma_{23} = \sigma_{23}/G \quad (4) \qquad G = \frac{E}{2(1+\nu)}$$

σ_{13} ← orthogonal plane to x_3 axis

direction of stress

For 2-D representation, let (1, 2) be the plane of interest with the 3 axis perpendicular to this plane, eg. Tunnel



Plane strain : By definition; • The (1, 2) plane is a 'principal' plane on which no shear stresses act

$$\therefore \quad \sigma_{13} = \sigma_{23} = 0$$

$$\longrightarrow \quad \gamma_{13} = \gamma_{23} = 0$$

• No displacement (strain) is allowed perpendicular to the (1, 2) plane

$$\therefore \quad \varepsilon_{33} = 0$$

Setting $\varepsilon_{33} = 0$ in equation (3)

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

Substituting into (1) and (2)

$$\varepsilon_{11} = \frac{1}{E}\left[(1-\nu^2)\sigma_{11} - \nu(1+\nu)\sigma_{22}\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left[(1-\nu^2)\sigma_{22} - \nu(1+\nu)\sigma_{11}\right]$$

or $\quad \underline{\varepsilon} = \underline{A}\,\underline{\sigma}$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} (1-\nu^2) & -\nu(1+\nu) & 0 \\ -\nu(1+\nu) & (1-\nu^2) & 0 \\ 0 & 0 & E/G \end{bmatrix}\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

Since for $\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$, $\underline{D} = \underline{A}^{-1}$

The third equation of the matrix identity is independent of the other 2 therefore $\rightarrow$ $\sigma_{12} = G\gamma_{12}$. The remaining 2×2 matrix may be inverted to give.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix}$$

N.B. since $\underline{D} = \underline{A}^{-1}$; $\underline{A}^{-1}\underline{D} = \underline{I}$ as a check. ✓

Plane stress

Definition ; • (1, 2) plane is principal plane $\therefore$ $\sigma_{13} = \sigma_{23} = 0$

• No stress perpendicular to (1, 2) plane $\sigma_{33} = 0$

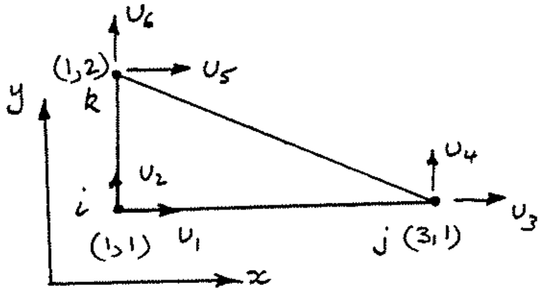
Substituting $\sigma_{13} = \sigma_{23} = 0$ into eqn(4) and $\sigma_{33} = 0$ into eqns (1, 2, 3) and rearranging terms:

$$\underline{\varepsilon} = \underline{A}\,\underline{\sigma} \qquad \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

$$\underline{\sigma} = \underline{D}\,\underline{\varepsilon} \qquad \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix}$$

Plane strain – most useful in geotechnical, geological situations
eg – slice through a dam, a tunnel – confined problems

Plane stress – many uses in structural mechanics – ie plate bending
fracture mechanics ie small specimens.



NUMERICAL EXAMPLE

Assume for simplicity that

$E = 1$; $\nu = 0.0$ (compressible, cork)

$\Delta = 1.0$

'D' matrix from eqn (19)

$$\underline{\underline{D}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad (1)$$

'a' matrix from eqn (18)

Area of triangular elt $\Delta = \cancel{2}\ 1.0$

$$\beta_{21} = y_j - y_k = -1$$
$$\beta_{22} = y_k - y_i = 1$$
$$\beta_{23} = y_i - y_j = 0$$
$$\beta_{31} = x_k - x_j = -2$$
$$\beta_{32} = x_i - x_k = 0$$
$$\beta_{33} = x_j - x_i = 2$$

$$\underline{a} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 1 & 2 & 0 \end{bmatrix} \qquad (2)$$

Note that in this case

$$K = \int_V \underline{a}^T \underline{D}\, a \, dV = \underline{a}^T \underline{D}\, \underline{a} \int_V dV \qquad (3)$$

$$K = \underline{a}^T \underline{D}\, \underline{a}\, t\, \Delta \qquad (4)$$

since $\underline{a}$ and $\underline{D}$ matrices <u>constants</u> where t = element thickness

Δ = element area

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} = \frac{t\Delta}{4} \begin{bmatrix} 3 & 1 & -1 & -1 & -2 & 0 \\ & 4\frac{1}{2} & 0 & -\frac{1}{2} & -1 & -4 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1}{2} & 1 & 0 \\ & \text{Symmetric} & & & 2 & 0 \\ & & & & & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \qquad (5)$$

Note the characteristic features:

- Positive definite (leading diagonal terms positive and non-zero)
- Symmetric about leading diagonal.

4:5 Coupled Hydro-Mechanical Models [6:2]

https://youtu.be/ve1EGEOT97k

[6:2] Linked Mechanisms

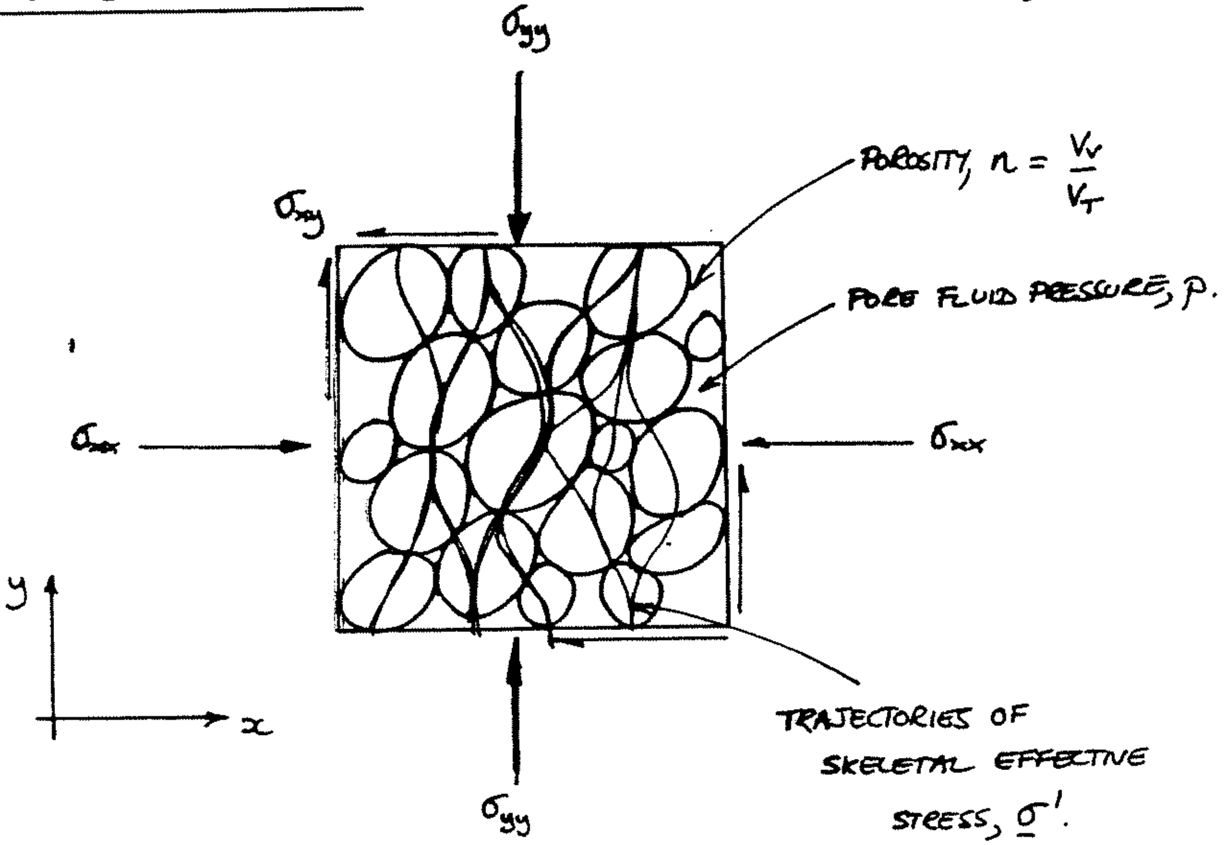
HM – Poromechanics

- Effective stresses
- FE equations
- Summary equations (Biot, 1941)
- EGEEfem
- Comsol

PROCESS COUPLINGS [T-H-M-C]

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & R_{22} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{p} \\ \underline{T} \\ \underline{c} \end{Bmatrix} + \begin{bmatrix} S_{11} & S_{12} & \cdots & \cdots \\ S_{21} & S_{22} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{Bmatrix} \dot{\underline{u}} \\ \dot{\underline{p}} \\ \dot{\underline{T}} \\ \dot{\underline{c}} \end{Bmatrix} = \begin{Bmatrix} \dot{\underline{f}} + \ldots \\ \underline{q}_F + \ldots \\ \underline{q}_T + \ldots \\ \underline{q}_M + \ldots \end{Bmatrix}$$

POROELASTIC RESPONSE M.A. BIOT (1941)



TOTAL STRESS = EFFECTIVE STRESS + PORE PRESSURE

Figure 4.1.1 Saturated porous medium

POROELASTIC EQUATIONS

EQUILIBRIUM (Virtual work)

Terzaghi's Law of effective stresses $\quad \partial\underline{\sigma} = \partial\underline{\sigma}' + \underline{m}\partial p \quad (1)$

$$\partial\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{Bmatrix} = \partial\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{Bmatrix}' + \underbrace{\begin{Bmatrix}1\\1\\0\end{Bmatrix}}_{\underline{m}}\partial p \quad (2)$$

Constitutive law $\quad \partial\underline{\sigma}' = \underline{E}\,\partial\underline{\varepsilon} \quad (3) \quad$ (note $\underline{E} \equiv \underline{D}$)

Substitute (3) into (1) gives

$$\partial\underline{\sigma} = \underline{E}\,\partial\underline{\varepsilon} + \underline{m}\,\partial p \quad (4)$$

Recall:
$VW_I = VW_E$
$\int \underline{\varepsilon}^T\underline{\sigma}\,dV = \underline{u}^T\underline{f}$
$\int (\underline{a}\underline{u})^T\underline{\sigma}\,dV = \underline{u}^T\underline{f}$
$\int \underline{u}^T\underline{a}^T\underline{\sigma}\,dV = \underline{u}^T\underline{f}$

From previous, equilibrium may be stated as: $\quad \int_A \underset{\underline{a}^T}{\underline{A}^T}\,\partial\underline{\sigma}\,dx\,dy = \partial\underline{f} \quad (5)$

Substituting (4) into (5) gives

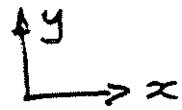
$$\int_A \underline{A}^T\underline{E}\,\underset{\partial\underline{\varepsilon} = \underline{A}\,\partial\underline{u}}{\underline{A}}\,dx\,dy\,\partial\underline{U} + \int_A \underline{A}^T\underline{m}\,\partial p\,dx\,dy = \partial\underline{f} \quad (6)$$

and if $p = \underline{b}\,\underline{p}$ then replacing in (6)

$$\boxed{\int_A \underline{A}^T\underline{E}\,\underline{A}\,dx\,dy\;\partial\dot{\underline{u}} + \int_A \underline{A}^T\underline{m}\,\underline{b}\,dx\,dy\;\partial\dot{\underline{p}} = \partial\dot{\underline{f}}} \quad (7)$$

And divide through by time, Δt.

$$\underline{B}_{11}\dot{\underline{U}} + \underline{B}_{12}\dot{\underline{p}} = \dot{\underline{f}} \quad (8)$$



CONSERVATION OF MASS

Darcy's Law:

$$\left\{\begin{matrix} v_x \\ v_y \end{matrix}\right\} = -\frac{k}{\mu}\left\{\begin{matrix} \partial/\partial x \\ \partial/\partial y \end{matrix}\right\}\underbrace{(p + \gamma y)}_{h\gamma_w} \qquad (9)$$

$$\underline{v} = -\frac{k}{\mu}\,\underline{\nabla}(p + \gamma y) \qquad (10)$$

Conservation of mass:

$$\underline{\nabla}^T \underline{v} = \underline{m}^T \underline{\dot{\varepsilon}} - \frac{n}{K_f}\dot{p} \qquad (11)$$

$$\underline{m}^T\underline{\dot{\varepsilon}} = (\dot{\varepsilon}_x + \dot{\varepsilon}_y)$$

Substituting (10) into (11)

$$-\underline{\nabla}^T \frac{k}{\mu}\underline{\nabla}(p + \gamma y) = \underline{m}^T\underline{\dot{\varepsilon}} - \frac{n}{K_f}\dot{p} \qquad (12)$$

or, more familiarly

$$\underbrace{-\frac{k}{\mu}\left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right]}_{\text{diffusion}} = \frac{\partial}{\partial t}\underbrace{(\dot{\varepsilon}_x + \dot{\varepsilon}_y)}_{\text{volume strain}} - \underbrace{\frac{n}{K_f}\frac{\partial p}{\partial t}}_{\text{fluid strain}} \qquad (13)$$

Apply Galerkin weighting:

$$\int_A w\Big[-\underline{\nabla}^T\frac{k}{\mu}\underline{\nabla}\,\overbrace{\underline{b}\,\underline{p}}^{p = \underline{b}\underline{p}} - \underline{m}^T\overbrace{\underline{A}\,\underline{\dot{u}}}^{\underline{\dot{\varepsilon}} = \underline{A}\underline{\dot{u}}} + \frac{n}{K_f}\underline{b}\,\underline{\dot{p}}\Big]\,dx\,dy = 0 \qquad (14)$$

Integrate in usual manner with, $w = b$, to give

$$\boxed{\begin{aligned} -\int_A \underline{a}^T \underline{k}\,\underline{a}\,dx\,dy\;\underline{p} + \int_A \underline{b}^T\underline{m}^T\underline{A}\,dx\,dy\;\underline{\dot{u}} \\ - \frac{n}{K_f}\int_A \underline{b}^T\underline{b}\,dx\,dy\;\underline{\dot{p}} = \underline{q} \end{aligned}} \qquad (15)$$

with $\begin{Bmatrix} \partial b/\partial x \\ \partial b/\partial y \end{Bmatrix} = \underline{a}$ (16)

and $\underline{K} = \frac{k}{\mu}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (17)

Symbolically this may be written (15) as

$$\underline{C}_{22}\,\underline{p} - \underline{B}_{22}\,\underline{\dot{p}} + \underline{B}_{21}\,\underline{\dot{u}} = \underline{q} \qquad (18)$$

COMBINING MATRIX EQUATIONS – CONSERVATION OF MOMENTUM
– CONSERVATION OF MASS

$$\begin{bmatrix} 0 & 0 \\ 0 & -\underline{C}_{22} \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{p} \end{Bmatrix}_\tau + \begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} \\ \underline{B}_{21} & \underline{B}_{22} \end{bmatrix} \begin{Bmatrix} \underline{\dot{u}} \\ \underline{\dot{p}} \end{Bmatrix}_\tau = \begin{Bmatrix} \underline{\dot{f}} \\ \underline{q} \end{Bmatrix}_\tau \qquad (19)$$

note $\underline{B}_{12} = \underline{B}_{21}^T$ $\therefore$ symmetric.

Defining time derivatives as:

$$\left.\begin{aligned} \underline{\dot{u}}_\tau &= \frac{1}{\Delta t}(\underline{u}_{t+\Delta t} - \underline{u}_t) \\ \underline{\dot{p}}_\tau &= \frac{1}{\Delta t}(\underline{p}_{t+\Delta t} - \underline{p}_t) \end{aligned}\right\} \qquad (20)$$

and assuming, for simplicity, $\lambda = 1.0$

$\therefore \tau = t + \Delta t$

Then substituting (20) into (19) gives

$$\boxed{\frac{1}{\Delta t}\begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} \\ \underline{B}_{21} & (\underline{B}_{22} + \Delta t\, \underline{C}_{22}) \end{bmatrix} \begin{Bmatrix} u \\ p \end{Bmatrix}_{t+\Delta t} = \begin{Bmatrix} \underline{\dot{f}} \\ \underline{q} \end{Bmatrix}_{t+\Delta t} + \frac{1}{\Delta t}\begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} \\ \underline{B}_{21} & \underline{B}_{22} \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{p} \end{Bmatrix}_t} \qquad (21)$$

or $\underline{K}^* \underline{h}_{t+\Delta t} = \underline{q}^*_{t+\Delta t}$

FINITE ELEMENT METHOD

Biot equations give
- Constitutive law and equilibrium equations
- Flow equation

Deformation

$$G\nabla^2 u_{x_i} + \frac{G}{(1-2\nu)}\frac{\partial \varepsilon_v}{\partial x} - \alpha\frac{\partial p}{\partial x} = b \qquad \text{3 equns.}$$

Divide through by $\partial/\partial t$

$$G\nabla^2 \frac{\partial u}{\partial t}\nearrow \dot{u} + \frac{G}{(1-2\nu)}\frac{\partial^2}{\partial x^2} u\, \cancel{\frac{\partial u}{\partial t}}\nearrow \dot{u} - \alpha\frac{\partial}{\partial x}\cancel{\frac{\partial p}{\partial t}}\nearrow \dot{p} = b$$

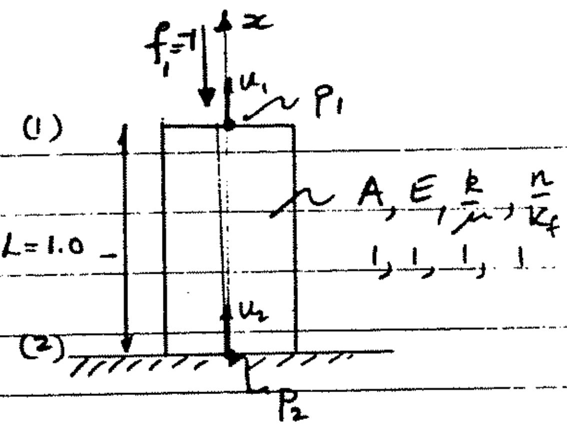
Flow equation

$$k\nabla^2 p = \frac{\partial}{\partial x}\alpha\cancel{\frac{\partial u}{\partial t}}\nearrow \dot{u} + \frac{1}{Q}\cancel{\frac{\partial p}{\partial t}}\nearrow \dot{p}$$

Matrix equations

$$\begin{bmatrix} 0 & 0 \\ 0 & \underline{E} \end{bmatrix}\begin{Bmatrix} \underline{u} \\ \underline{p} \end{Bmatrix}^t + \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{D} \end{bmatrix}\begin{Bmatrix} \dot{u} \\ \dot{p} \end{Bmatrix}^t = \begin{Bmatrix} b \\ q \end{Bmatrix}^t$$

EXAMPLE 4.1



Consider a two noded one-dimensional element – as shown

Using similar terminology

$$\varepsilon_x = \frac{1}{L}[1 \; ; \; -1]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \underline{\varepsilon} = \underline{A} \cdot \underline{u} \qquad (1)$$

$$p = \left[\frac{x}{L} \; ; \; \left(1 - \frac{x}{L}\right)\right]\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} \qquad p = \underline{b}\,\underline{p} \qquad (2)$$

$$\frac{\partial p}{\partial x} = \frac{1}{L}[1 \; ; \; -1]\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} \qquad \frac{\partial p}{\partial x} = \underline{a}\,\underline{p} \qquad (3)$$

$$\underline{m} = [1] \quad ; \quad \underline{E} = E \qquad (4)$$

Matrices

$$\underline{B}_{11} = \int_A \underline{A}^T \underline{E}\, \underline{A}\, dx\, dy = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (5)$$

$$\underline{B}_{22} = \frac{n}{K_f}\int_A \underline{b}^T \underline{b}\, dx\, dy = \frac{n}{K_f}\frac{AL}{8}\begin{bmatrix} 8/3 & 4/3 \\ 4/3 & 8/3 \end{bmatrix} \qquad (6)$$

$$\text{or lumped} = \frac{n}{K_f}\frac{AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (7)$$

$$\underline{C}_{22} = \frac{k}{\mu}\int_A \underline{a}^T \underline{a}\, dx\, dy = \frac{k}{\mu}\frac{A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (8)$$

$$\underline{B}_{12} = \underline{B}_{21}^T = \int_A \underline{A}^T \underline{m}\ \underline{b}\ dx\,dy$$

$$\underline{B}_{12} = A\int_A \frac{1}{L}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}[1]\left\{ \frac{x}{L}\ ;\ (1-\frac{x}{L}) \right\} dx$$

$$\int_0^L x/_L\,dx = \left[\frac{x^2}{2L}\right]_0^L = \cancel{\frac{1}{2L}}\left(\frac{L}{2}\right)$$

$$\underline{B}_{12} = \frac{A}{L}\int_0^L \begin{bmatrix} x/_L & (1-x/_L) \\ -x/_L & -(1-x/_L) \end{bmatrix} dx$$

$$\int_0^L (1-\frac{x}{L})\,dx = \left[x - \frac{x^2}{2L}\right]_0^L = \cancel{(1-\frac{1}{2L})}\ \left(L - \frac{L}{2}\right)$$

$L = 1.0$

$$\underline{B}_{12} = \frac{A}{2\cancel{L}}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad \cancel{\frac{L}{2}\frac{A}{L}} = \frac{A}{2} \qquad (9)$$

and $\underline{B}_{21} = \underline{B}_{12}^T$

$$\cancel{\frac{A}{L}\left(L - \frac{L}{2}\right) = A}$$

$$A\left(1 - \frac{1}{2}\right) = \left(\frac{1}{2}\right)A$$

Transcient Solution

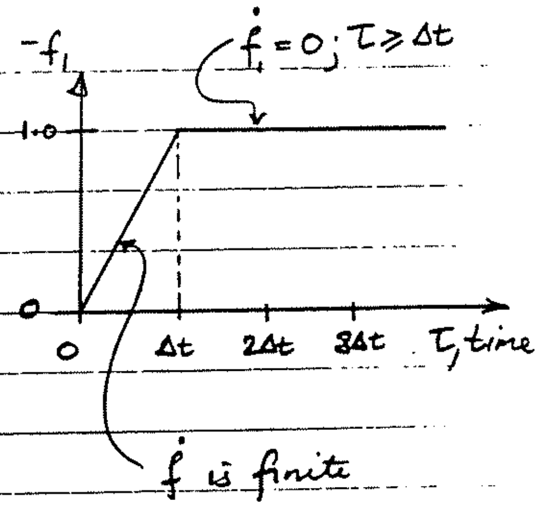
$$\frac{1}{\Delta t}\begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} \\ \underline{B}_{21} & \underline{B}_{22} + \Delta t\,\underline{C}_{22} \end{bmatrix}\begin{Bmatrix} \underline{u} \\ \underline{p} \end{Bmatrix}_{t+\Delta t} = \begin{Bmatrix} \dot{\underline{f}} \\ \underline{q} \end{Bmatrix} + \frac{1}{\Delta t}\begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} \\ \underline{B}_{21} & \underline{B}_{22} \end{bmatrix}\begin{Bmatrix} \underline{u} \\ \underline{p} \end{Bmatrix}_t \qquad (10)$$

Setting $\Delta t = 1.0$, for one element then

$$\begin{bmatrix} 1 & -1 & \frac{1}{2} & \frac{1}{2} \\ -1 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & a_{11} & \frac{k}{\mu} \\ \frac{1}{2} & -\frac{1}{2} & \frac{k}{\mu} & a_{11} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 = 0 \\ p_1 \\ p_2 \end{Bmatrix}_{t+\Delta t} = \begin{bmatrix} 1 & -1 & \frac{1}{2} & \frac{1}{2} \\ -1 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & a_{12} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & a_{12} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ p_1 \\ p_2 \end{Bmatrix}_t + \begin{Bmatrix} \dot{f}_1 \\ \dot{f}_2 \\ q_1 \\ q_2 \end{Bmatrix}$$

$$a_{11} = -\frac{k}{\mu}\Delta t - \frac{n}{K_f}\frac{AL}{2} \qquad\qquad a_{12} = -\frac{n}{K_f}\frac{AL}{2}$$

The boundary conditions may be prescribed as changes in total stresses by the application of a vertical force, f_1, at time $0 + \frac{1}{2}\Delta t$. This must be distributed as shown right.



Rate of force application is then

$$\dot{\underline{f}} = \frac{1}{\Delta t}\left[\underline{f}_{t+\Delta t} - \underline{f}_t\right]$$

<u>To solve equation system</u>

1. The magnitudes of $\dot{\underline{f}}$ are known at all times (prescribed)

2. Prescribe other boundary conditions

$$f_1 = 0 \ @\ \tau = 0 \qquad f_1 = -1.0 \ @\ \tau = \Delta t$$
$$u_2 = 0 \qquad \tau \geqslant 0$$
$$p_1 = 0 \qquad \tau \geqslant 0$$

Recall basic forms of equations:

$$\frac{\partial^2 H}{\partial x^2} \longrightarrow \int \underline{a}^T \underline{D} \underline{a} \, dV \, \underline{H}$$

$$v \frac{\partial H}{\partial x} \longrightarrow \int \underline{b}^T \underline{v} \, \underline{a} \, \partial V \, \underline{H}$$

$$\frac{\partial H}{\partial t} \equiv \dot{H} \longrightarrow \int \underline{b}^T \underline{b} \, dV \, \dot{H}$$

1-D Biot eqns

$$G \frac{\partial^2 u}{\partial x^2} + \frac{G}{(1-2\nu)} \frac{\partial \varepsilon}{\partial x} - \alpha \frac{\partial p}{\partial x} = 0$$

$$k \frac{\partial^2 p}{\partial x^2} = \alpha \frac{\partial \overset{\frac{\partial u}{\partial x}}{\varepsilon}}{\partial t} + \frac{1}{Q} \frac{\partial p}{\partial t}$$

Second equation.

$$k \frac{\partial^2 p}{\partial x^2} = \alpha \frac{\partial}{\partial t} \frac{\partial u}{\partial x} + \frac{1}{Q} \frac{\partial p}{\partial t}$$

$$k \frac{\partial^2 p}{\partial x^2} = \alpha \frac{\partial \dot{u}}{\partial x} + \frac{1}{Q} \dot{p}$$

$$\int \underline{a}^T \underline{D} \underline{a} \, dV \, \underline{p} \pm \int \underline{b}^T \underline{\alpha} \, \underline{a} \, dV \, \underline{\dot{u}} \pm \frac{1}{Q} \int \underline{b}^T \underline{b} \, dV \, \underline{\dot{p}} = q$$

$$\int \underline{b}^T \underline{m} \, \underline{a} \, dV \, \underline{\dot{u}}$$

SUMMARY

EQUILIBRIUM

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = b_x - \rho \frac{\partial^2 u_x}{\partial t^2}$$

(1.2) 3 equns

$$\varepsilon_x = \frac{\partial u_x}{\partial x} \; ; \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \; ; \quad \Theta = \Delta V_f / V$$

CONSTITUTIVE

$$\sigma_x = 2G\left(\varepsilon_x + \nu \frac{\varepsilon_v}{1-2\nu}\right) - \alpha p$$

(2.11) (3 eqs)

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] + \frac{p}{3H}$$

(2.4) (3 eqs)

$$\tau_{xy} = G\,\gamma_{xy}$$

(3 eqs)

$$\gamma_{xy} = \tau_{xy}/G$$

(3 eqs)

$$p = (\Theta - \alpha \varepsilon_v) Q$$

(2.12) (1 eq)

$$\Theta = \frac{1}{3H}(\sigma_x + \sigma_y + \sigma_z) + \frac{p}{R}$$

(1 eq)

(2.10)

$$\Theta = \alpha \varepsilon_v + \frac{p}{Q}$$

(2.12)

Evaluate parameters: (E, ν, H, R)

E & ν from (2.4) $\longrightarrow G$

H & R from (2.10)

Then $\alpha = \dfrac{2(1+\nu)}{3(1-2\nu)} \dfrac{G}{H}$; $\dfrac{1}{Q} = \dfrac{1}{R} - \dfrac{\alpha}{H}$

FLOW

$$\Theta = \alpha \varepsilon_v + \frac{p}{Q} \qquad v_x = -k\frac{dp}{dx}$$

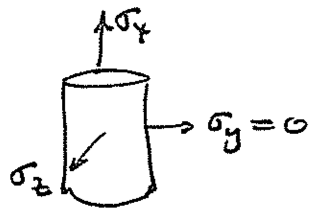
$$\frac{\partial \Theta}{\partial t} = -\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} - \frac{\partial v_z}{\partial z}$$

Substitute (2.11) into (1.2) (3 equns) $\quad G\nabla^2 u_x + \dfrac{G}{(1-2\nu)} \dfrac{\partial \varepsilon_v}{\partial x} - \alpha \dfrac{\partial p}{\partial x} = 0$

Substitute for flow: $\quad k\nabla^2 p = \dfrac{1}{Q}\dfrac{\partial p}{\partial t} + \alpha \dfrac{\partial \varepsilon_v}{\partial t}$

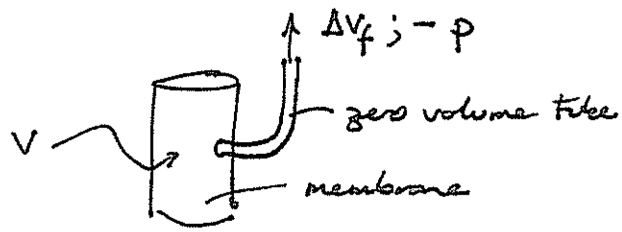
PHYSICAL INTERPRETATION OF PARAMETERS $\quad E, \nu, H, R \rightarrow G, Q, \alpha$

Dry:



$$E = \frac{\sigma_x}{\varepsilon_x} \quad ; \quad \varepsilon_z = \varepsilon_y = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

Fluid filled + jacketed:



No change in total stress

1. Apply $-p$ and remove ΔV_f

$$\theta = \frac{\Delta V_f}{V} \; ; \quad \theta = \frac{1}{3H}\cancelto{0}{(\sigma_x + \sigma_y + \sigma_z)} + \frac{P}{R}$$

$$\therefore \quad R = P/\theta$$

(fluid strain with change in effective stress).

2. Measure ΔV_s (volume change of soil) for applied $-p$ and ΔV_f

$$\varepsilon_V = \varepsilon_x + \varepsilon_y + \varepsilon_z = \cancelto{0}{\frac{3\sigma_m}{E}}(1-2\nu) + \frac{P}{H} \qquad \therefore \quad H = \frac{P}{\varepsilon_V}$$

(solid strain with effective stress)

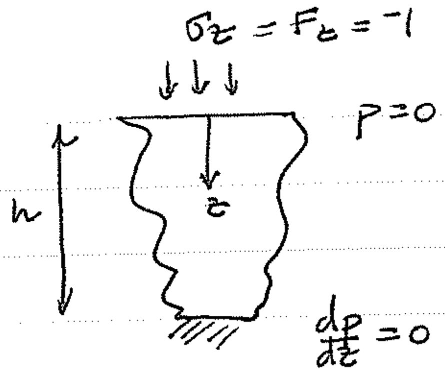
From: $\quad E, \nu, H, R$

$$G = E/2(1+\nu)$$

$$\alpha = \frac{2(1+\nu)}{3(1-2\nu)}\frac{G}{H}$$

$$\frac{1}{Q} = \frac{1}{R} - \frac{\alpha}{H}$$

Validation



$$E = 1 \ (Pa), \quad \nu = 0 \quad \Big\} \ G = 1/2$$

$$D = k/\mu = 10^{-3} \ m^2/Pa.s$$

$$\alpha = 1$$

$$1/Q = 10^{-5} \ Pa^{-1}$$

Initial Pressure, p_i:

$$\Theta = \alpha \varepsilon_v + \frac{p}{Q} \quad \therefore \quad \varepsilon_v = \frac{\partial u_z}{\partial z} = -\frac{1}{\alpha}\frac{p}{Q}$$

$$\sigma_z = 2G\left(\varepsilon_z + \frac{\nu}{(1-2\nu)}(\cancel{\varepsilon_x}^{0} + \cancel{\varepsilon_y}^{0} + \varepsilon_z)\right) - \alpha p$$

$$\sigma_z = 2G\frac{(1-\nu)}{(1-2\nu)}\varepsilon_z - \alpha p$$

$$-\frac{\sigma_z}{p} = 2G\frac{(1-0)}{(1-2\nu)}\frac{1}{\alpha Q} + \alpha \longrightarrow 0 + 1$$

$$\therefore \ p = -\sigma_z$$

Time to 50% consolidation, t_D^{50}:

$$\frac{k}{\mu}\nabla^2 p = \frac{1}{Q}\frac{\partial p}{\partial t} + \alpha\frac{\partial \varepsilon_v}{\partial t}$$

For $\frac{\partial \sigma}{\partial t} \equiv 0$: $\quad \varepsilon_v = \frac{p}{3H}$ and $H = \frac{2(1+\nu)}{3(1-2\nu)}\frac{G}{\alpha}$

$$\underbrace{\frac{k}{\mu}\left(\frac{1}{Q} + \frac{\alpha^2}{G}\frac{(1-2\nu)}{2(1+\nu)}\right)^{-1}}_{c}\nabla^2 p = \frac{\partial p}{\partial t} \quad \Rightarrow \quad c\frac{\partial^2 p}{\partial z^2} = \frac{\partial p}{\partial t}$$

$$c = \left(10^{-5} + \frac{2}{1}\cdot\frac{1}{2}\right)10^{-3} = \overset{1.0}{\cancel{0.5}} \times 10^{-3}.$$

125s ~~256s~~

50% complete $t_D^{50} = 0.2$; $\quad 0.2 = \frac{ct}{h^2} \quad \therefore \quad \frac{0.2h^2}{c} = t = \frac{0.2(0.8)^2}{1.0\cancel{0.5} \times 10^{-3}} =$