Earth 202: Homework #5 (Stiff equations and Euler's/RK methods)

Due: Thurs., Mar. 6

Suppose we have a system of two ordinary differential equations for two functions $u(t)$ and $v(t)$

$$du/dt = 998\,u + 1998\,v \qquad u(0) = 1$$

$$dv/dt = -999\,u - 1999\,v \qquad v(0) = 0$$

1. Find the analytic solution to this system and compute the values of u and v at $t = 1$. (Hint: Try out the substitution: $y = u + v$, $z = u + 2v$). To make life somewhat easier, and to allow you to build the STELLA model without first solving the analytic problem, here are the answers:

   $$u(t) = 2e^{-t} - e^{-1000t} \qquad v(t) = e^{-1000t} - e^{-t}$$

2. Compute $u(1)$ and $v(1)$ numerically using the forward Euler method. Divide the interval ($0 < t < 1$) into N equal steps and find solutions for N = 8000, 4000, 2000, 1000, 505, and 500. What happens if N < 500? Using STELLA, calculate the relative error at time 1 for each value of N. (Don't do this by hand—build it into your program!) How does the error vary with Δt (=1/N)? Is there some limiting value of N beyond which accuracy does not improve? Why? Define the error as

   $$\text{Error} = \log_{10}\{\{[(u-u_0)/u_0]^2 + [(v-v_0)/v_0]^2\}^{0.5} + 1.\text{e-}60\} \quad \text{at time } t = 1$$

   where (u_0, v_0) represents the analytic solution. This is just a root-mean-square error expressed as a logarithm to the base 10.

3. Compute $u(1)$ and $v(1)$ using the 4^{th}-order Runge-Kutta method. Use the same values for N as before, along with N= 400 and 350. How does the accuracy depend on Δt in this case? As before, is there a number N beyond which the solution does not improve?