EARTH 202: Stiff equations/Explicit and implicit methods

Due: Thursday, Apr. 5

Suppose we have a system of two ordinary differential equations for two functions $u(t)$ and $v(t)$

$$du/dt = 998\,u + 1998\,v \qquad u(0) = 1$$

$$dv/dt = -999\,u - 1999\,v \qquad v(0) = 0$$

1. Find the analytic solution to this system and compute the values of u and v at $t = 1$. (Hint: Try out the substitution: $y = u + v$, $z = u + 2v$)

2. Compute $u(1)$ and $v(1)$ numerically (in STELLA) using the forward Euler method. Find solutions for $\Delta t = 0.000125$, 0.00025, 0.0005, 0.001, 0.0019, 0.002. These correspond to dividing the interval 0-1 into N equal steps, with N = 8000, 4000, 2000, 1000, 526, and 500. What happens if $\Delta t > 0.002$ (i.e., $N < 500$)? Calculate the relative error in $u(1)$ for each value of Δt. (Don't do this by hand—build it into your program!) How does the error vary with Δt (=1/N)? Is there some limiting value of N beyond which accuracy does not improve? Define the error as

$$\text{Error} = \{[(u-u_0)/u_0]^2 + [(v-v_0)/v_0]^2\}^{0.5} \quad \text{at time } t = 1$$

   where (u_0, v_0) represents the analytic solution.

3. Compute $u(1)$ and $v(1)$ using the 2nd and 4th-order Runge-Kutta methods in STELLA. Use the same time steps as before. How does the accuracy depend on Δt in this case?

4. Compare your answers with the solutions generated using the (implicit) reverse Euler method and the (semi-implicit) midpoint method. We will generate these in class by running a Fortran program written by one of your professors. For this exercise, we will use the same values for N as before, along with N= 100 and 10. Show that these methods allow one to bypass the stability criteria that apply to the (explicit) forward Euler and Runge-Kutta methods.