Sources of error

In general, there are two different sources of error in solving differential equations numerically:

1) Truncation error

This comes from using only a finite number of terms in the (infinite) Taylor series expansion. For example, in forward Euler we use only the first two terms:

$$\frac{dx(t)}{dt} = F(x,t)$$

$$x_{n+1} = x_n + \frac{dx}{dt}\Big|_{t=t_n}(t_{n+1}-t_n) \underbrace{+ \frac{1}{2}\frac{d^2x}{dt^2}\Big|_{t=t_n}(t_{n+1}-t_n)^2}_{\text{neglect}}$$

$$= x_n + F(x_n,t_n)\,\Delta t + \underbrace{O(\Delta t^2)}_{\text{truncation error}}$$

2) Roundoff error

The computer can only keep track of so many significant digits (about 8 for a 32-bit machine, or 15 for a 64-bit machine). Thus, at each step, we make an error in either the 9^{th} or 16^{th} decimal place.

Eq: $\frac{dx}{dt} = F(x,t)$

Accuracy of Euler's methods:

Forward: $$x_{n+1} = x_n + \frac{dx}{dt}\Big|_{t=t_n}(t_{n+1}-t_n) + \underbrace{\frac{1}{2}\frac{d^2x}{dt^2}\Big|_{t=t_n}(t_{n+1}-t_n)^2}_{\text{neglect}}$$

$$x_{n+1} \cong x_n + F(x_n,t_n)\,\Delta t + O(\Delta t^2)$$

Reverse: $$x_{n+1} = x_n + F(x_{n+1},t_{n+1})\,\Delta t + O(\Delta t^2)$$

Crank-Nicholson: $$x_{n+1} = x_n + \frac{1}{2}\left[F(x_n,t_n) + F(x_{n+1},t_{n+1})\right]\cdot\Delta t$$

* Note: Deriving the accuracy of a method is a bit technical. Heuristically, though, a method in which the truncation error is $O(\Delta t^2)$ is first order (i.e., the error decreases linearly with Δt), because the number of steps is $\propto 1/\Delta t$. Thus, the cumulative error goes as

$$\text{Error} \propto O(\Delta t^2)\cdot\frac{1}{\Delta t} \propto \Delta t$$

for a first-order method.

Runge-Kutta methods.

Put these in the same notation as Euler's method.

1) Second-order method:

Let

$$\Delta x_1 = F(x_n, t_n)\,\Delta t$$
$$x_{n+1/2} = x_n + \tfrac{1}{2}\Delta x_1$$
$$t_{n+1/2} = t_n + \tfrac{1}{2}\Delta t$$
$$\Delta x_2 = F(x_{n+1/2}, t_{n+1/2})$$
$$x_{n+1} = x_n + \Delta x_2 + O(\Delta t^3)$$

2) 4^{th}-order method

$$\Delta x_1 = F(x_n, t_n)\,\Delta t$$
$$x_1 = x_n + \tfrac{1}{2}\Delta x_1$$
$$t_{n+1/2} = t_n + \tfrac{1}{2}\Delta t$$
$$\Delta x_2 = F(x_1, t_{n+1/2})\,\Delta t$$
$$x_2 = x_n + \tfrac{1}{2}\Delta x_2$$
$$\Delta x_3 = F(x_2, t_{n+1/2})\,\Delta t$$
$$x_3 = x_n + \tfrac{1}{2}\Delta x_3$$
$$t_{n+1} \equiv t_n + \Delta t$$
$$\Delta x_4 = F(x_3, t_{n+1})\,\Delta t$$

$$x_{n+1} = x_n + \frac{\Delta x_1}{6} + \frac{\Delta x_2}{3} + \frac{\Delta x_3}{3} + \frac{\Delta x_4}{6} + O(\Delta t^5)$$

Types of errors:

1) Truncation error
2) Roundoff error

Now, talk about stability $\Rightarrow$ implicit methods

Forward Euler: Stability

Recall that the method comes from a Taylor series expansion;

$$\frac{dx}{dt} = F(x_n, t_n) \qquad x(t_0) = x_0$$

$$x_{n+1} \cong x_n + F(x_n, t_n)\Delta t + \underbrace{\frac{1}{2}F'(x_n, t_n)(\Delta t)^2}_{\text{error}}$$

Now, consider the exponential function:

$$\frac{dx}{dt} = \lambda x(t) \qquad x(t_0) = x_0$$

This has the analytic solution:

$$x(t) = x_0 e^{\lambda t} \qquad F(x_n) = \lambda x_n$$
$$F'(x_n) = \lambda^2 x_n$$

so

$$x_{n+1} = x_n + \lambda x_n \Delta t + \frac{1}{2}\lambda^2 x_n (\Delta t)^2$$

$$= x_n\left(1 + \lambda \Delta t + \underbrace{\frac{1}{2}(\lambda \Delta t)^2}_{\text{error}}\right)$$

We run into trouble if the error term is larger than the first-order term, i.e. if

$$\frac{1}{2}(\lambda \Delta t) > 1$$

or

$$\Delta t > \frac{2}{\lambda}$$

So, the requirement for stability is just the opposite of this:

$$\boxed{\Delta t < \frac{2}{\lambda}}$$

Now, consider a decaying exponential:

$$x(t) = x_0 e^{-\lambda t}$$

$$F'(x_n) = -\lambda x_n$$
$$F''(x_n) = \lambda^2 x_n$$

$$x_{n+1} = x_n - \lambda x_n \Delta t + \frac{1}{2} \lambda^2 x_n (\Delta t)^2$$

$$= x_n \left(1 - \lambda \Delta t + \frac{1}{2} (\lambda \Delta t)^2\right)$$

We still run into trouble if

$$\frac{1}{2} (\lambda \Delta t) > 1$$

or

$$\boxed{\Delta t > \frac{2}{\lambda}}$$

More rigorous stability analysis

1) $\frac{dx}{dt} = -\lambda x(t)$ $\qquad x(t) = Ae^{-\lambda t}$

Let $A=1$. Use Forward Euler method:

2) $\frac{x_{n+1} - x_n}{\Delta t} = -\lambda x_n$

$$x_{n+1} = x_n - \lambda x_n \Delta t$$
$$= \underbrace{(1 - \lambda \Delta t)}_{\xi} x_n$$

$\xi \equiv$ amplification factor

For stability, we must have $|\xi| < 1$. Thus

3) $|1 - \lambda \Delta t| < 1$

so

$$\lambda \Delta t < 2 \Rightarrow \boxed{\Delta t < \frac{2}{\lambda}}$$

which is exactly what we got previously by considering the relative size of terms in the Taylor expansion.

Now, let's look at reverse Euler:

4) $\frac{x_{n+1} - x_n}{\Delta t} = -\lambda x_{n+1}$

$$x_{n+1} = x_n - \lambda \Delta t \, x_{n+1}$$

$$x_{n+1}(1+\lambda\Delta t) = x_n$$

or

$$x_{n+1} = \underbrace{\left(\frac{1}{1+\lambda\Delta t}\right)}_{\xi} x_n$$

so

$$\xi = \frac{1}{1+\lambda\Delta t} < 1 \quad \text{for all } \Delta t$$

$\Rightarrow$ unconditionally stable!

<u>Crank-Nicholson</u>

5) $$\frac{x_{n+1} - x_n}{\Delta t} = -\lambda\left(\frac{x_{n+1} + x_n}{2}\right)$$

$$x_{n+1} = x_n - \lambda\Delta t/2 \, (x_{n+1} + x_n)$$

$$x_{n+1}(1+\lambda\Delta t/2) = x_n(1-\lambda\Delta t/2)$$

$$x_{n+1} = x_n \underbrace{\left(\frac{1-\lambda\Delta t/2}{1+\lambda\Delta t/2}\right)}_{\xi}$$

$$\xi = \frac{1-\lambda\Delta t/2}{1+\lambda\Delta t/2} < 1 \quad \text{for all } \Delta t$$

$\Rightarrow$ also unconditionally stable...