

Sources of error

In general, there are two different sources of error in solving differential equations numerically:

1) Truncation error

This comes from using only a finite number of terms in the (infinite) Taylor series expansion. For example, in forward Euler we use only the first two terms:

$$\frac{dx(t)}{dt} = F(x, t)$$

$$x_{n+1} = x_n + \frac{dx}{dt} \Big|_{t=t_n} (t_{n+1} - t_n) + \underbrace{\frac{1}{2} \frac{d^2x}{dt^2} \Big|_{t=t_n} (t_{n+1} - t_n)^2}_{\text{neglect}}$$

$$= x_n + F(x_n, t_n) \Delta t + \underbrace{O(\Delta t^2)}_{\text{truncation error}}$$

2) Roundoff error

The computer can only keep track of so many significant digits (about 8 for a 32-bit machine, or 15 for a 64-bit machine). Thus, at each step, we make an error in either the 9th or 16th decimal place.

$$\text{Eq: } \frac{dx}{dt} = F(x, t)$$

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Accuracy of Euler's methods:

$$\text{Forward: } x_{n+1} = x_n + \left. \frac{dx}{dt} \Big|_{t=t_n} (t_{n+1} - t_n) \right. \\ + \underbrace{\frac{1}{2} \frac{d^2x}{dt^2} \Big|_{t=t_n} (t_{n+1} - t_n)^2}_{\text{neglect}}$$

$$x_{n+1} \cong x_n + F(x_n, t_n) \Delta t + O(\Delta t^2)$$

$$\text{Reverse: } x_{n+1} = x_n + F(x_{n+1}, t_{n+1}) \Delta t + O(\Delta t^2)$$

$$\text{Crank-Nicholson: } x_{n+1} = x_n + \frac{1}{2} \left[F(x_n, t_n) + F(x_{n+1}, t_{n+1}) \right] \cdot \Delta t$$

* Note: Deriving the accuracy of a method is a bit technical. Heuristically, though, a method in which the truncation error is $O(\Delta t^2)$ is first order (i.e., the error decreases linearly with Δt , because the number of steps is $\propto 1/\Delta t$). Thus, the cumulative error goes as

$$\text{Error} \propto O(\Delta t^2) \cdot \frac{1}{\Delta t} \propto \Delta t$$

for a first-order method.

Runge-Kutta methods

Put these in the same notation as Euler's method.

1) Second-order method:

Let

$$\Delta X_1 = F(X_n, t_n) \Delta t$$

$$X_{n+1/2} = X_n + \frac{1}{2} \Delta X_1$$

$$t_{n+1/2} = t_n + \frac{1}{2} \Delta t$$

$$\Delta X_2 = F(X_{n+1/2}, t_{n+1/2}) \Delta t$$

$$X_{n+1} = X_n + \Delta X_2 + O(\Delta t^3)$$

2) 4th-order method

$$\Delta X_1 = F(X_n, t_n) \Delta t$$

$$X_1 = X_n + \frac{1}{2} \Delta X_1$$

$$t_{n+1/2} = t_n + \frac{1}{2} \Delta t$$

$$\Delta X_2 = F(X_1, t_{n+1/2}) \Delta t$$

$$X_2 = X_n + \frac{1}{2} \Delta X_2$$

$$\Delta X_3 = F(X_2, t_{n+1/2}) \Delta t$$

$$X_3 = X_n + \frac{1}{2} \Delta X_3$$

$$t_{n+1} = t_n + \Delta t$$

$$\Delta X_4 = F(X_3, t_{n+1}) \Delta t$$

$$X_{n+1} = X_n + \frac{\Delta X_1}{6} + \frac{\Delta X_2}{3} + \frac{\Delta X_3}{3} + \frac{\Delta X_4}{6} + O(\Delta t^5)$$

Types of errors:

1) Truncation error

2) Roundoff error

Now, talk about stability \Rightarrow implicit methods

Forward Euler: Stability

Recall that the method comes from a Taylor series expansion:

$$\frac{dx}{dt} = F(x_n, t_n) \quad x(t_0) = x_0$$

$$x_{n+1} \approx x_n + F(x_n, t_n) \Delta t + \underbrace{\frac{1}{2} F'(x_n, t_n) (\Delta t)^2}_{\text{error}}$$

Now, consider the exponential function:

$$\frac{dx}{dt} = \lambda x(t) \quad x(t_0) = x_0$$

This has the analytic solution:

$$x(t) = x_0 e^{\lambda t}$$

$$F(x_n) = \lambda x_n$$

$$F'(x_n) = \lambda^2 x_n$$

so

$$x_{n+1} = x_n + \lambda x_n \Delta t + \frac{1}{2} \lambda^2 x_n (\Delta t)^2$$

$$= x_n \left(1 + \lambda \Delta t + \underbrace{\frac{1}{2} (\lambda \Delta t)^2}_{\text{error}} \right)$$

We run into trouble if the error term is larger than the first-order term, i.e. if

$$\frac{1}{2} (\lambda \Delta t)^2 > 1$$

or

$$\Delta t > \frac{2}{\lambda}$$

So, the requirement for stability is just the opposite of this:

$$\Delta t < \frac{2}{\lambda}$$

Now, consider a decaying exponential:

$$x(t) = x_0 e^{-\lambda t}$$

$$F(x_n) = -\lambda x_n$$

$$F'(x_n) = \lambda^2 x_n$$

$$x_{n+1} = x_n - \lambda x_n \Delta t + \frac{1}{2} \lambda^2 x_n (\Delta t)^2$$

$$= x_n \left(1 - \lambda \Delta t + \frac{1}{2} (\lambda \Delta t)^2 \right)$$

We still run into trouble if

$$\frac{1}{2} (\lambda \Delta t)^2 > 1$$

or

$$\Delta t > \frac{2}{\lambda}$$

More rigorous stability analysis

$$1) \frac{dx}{dt} = -\lambda x(t) \quad x(t) = Ae^{-\lambda t}$$

Let $A=1$, Use Forward Euler method:

$$2) \frac{x_{n+1} - x_n}{\Delta t} = -\lambda x_n$$

$$\begin{aligned} x_{n+1} &= x_n - \lambda x_n \Delta t \\ &= \underbrace{(1 - \lambda \Delta t)}_{\xi} x_n \end{aligned}$$

$\xi \equiv$ amplification factor

For stability, we must have $|\xi| < 1$,
Thus

$$3) |1 - \lambda \Delta t| < 1$$

so

$$\lambda \Delta t < 2 \Rightarrow \boxed{\Delta t < \frac{2}{\lambda}}$$

which is exactly what we got previously by considering the relative size of terms in the Taylor expansion.

Now, let's look at reverse Euler:

$$4) \frac{x_{n+1} - x_n}{\Delta t} = -\lambda x_{n+1}$$

$$x_{n+1} = x_n - \lambda \Delta t x_{n+1}$$

$$x_{n+1} (1 + \lambda \Delta t) = x_n$$

or

$$x_{n+1} = \underbrace{\left(\frac{1}{1 + \lambda \Delta t} \right)}_{\xi} x_n$$

so

$$\xi = \frac{1}{1 + \lambda \Delta t} < 1 \text{ for all } \Delta t$$

⇒ unconditionally stable!

Crank-Nicholson

$$5) \frac{x_{n+1} - x_n}{\Delta t} = -\lambda \left(\frac{x_{n+1} + x_n}{2} \right)$$

$$x_{n+1} = x_n - \lambda \Delta t / 2 (x_{n+1} + x_n)$$

$$x_{n+1} (1 + \lambda \Delta t / 2) = x_n (1 - \lambda \Delta t / 2)$$

$$x_{n+1} = x_n \underbrace{\left(\frac{1 - \lambda \Delta t / 2}{1 + \lambda \Delta t / 2} \right)}_{\xi}$$

$$\xi = \frac{1 - \lambda \Delta t / 2}{1 + \lambda \Delta t / 2} < 1 \text{ for all } \Delta t$$

⇒ also unconditionally stable...